ON THE THEORY OF LONG WAVES*

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A new method of studying plane steady wave motion of a gravity fluid is elucidated in this paper. This method succeeds in establishing the existence of a solitary wave, for example, and in giving the first complete foundation for the approximate Rayleigh theory [1], which concerns the theory of finite-amplitude long waves. Underlying the method are general boundary properties of univalent functions, used earlier by the author to construct a qualitative theory of jet fluid motions [2].

1. FORMULATION OF THE PROBLEM

As is known, the problem of the steady wave motion of a gravity fluid in a channel of variable depth reduces to such a boundary-value problem of conformal mapping theory: Let a line $\Gamma_0: y = y_0(x)$, where the function $y_0(x)$ is single-valued and continuous, together with its two derivatives for all values of x be given in the plane of the complex variable $z = x + iy$. Find the line Γ: $y = y(x)$, $y(x) > y_0(x)$ such that for the conformal mapping $\zeta = f(z, \Gamma_0, \Gamma)$, $\zeta = \xi + i\eta$ of the domain $D(\Gamma_0, \Gamma)$ bounded by Γ_0 and Γ, the relationship

$$I_v(\Gamma_0,\Gamma)=|f'(z,\Gamma)|^2- C+\lambda y=0, \tag{1}$$

would hold in the strip $v < \eta < h$, $f(\pm\infty, \Gamma_0, \Gamma) = \pm\infty$ along the line Γ, where C and λ are given constants. Hydrodynamically, the function f denotes the complex potential of a moving fluid, and the number h is the discharge, while (1) corresponds to a constant pressure on the free surface. If $y_0(x) = \text{const}$, then by imposing certain conditions on a motion with the potential f we obtain wave motion in a channel of finite depth with zero transverse fluid velocity.

Henceforth, let us limit ourselves to consideration of the case when h is sufficiently small and when the quantities $(1/h)|y_0(x)-h|$, $(1/h)|y(x)-h|$ together with the first three derivatives, as well as $|C-\xi|$, $|\lambda-(2/h)|$, are small together with h. In conformity with this, let us set up some relationships concerning the conformal mapping of the domains $D(\Gamma_0, \Gamma)$ close to the strip $0 < y < h$ onto the strip $0 < \eta < h$.

*The variational methods of conformal mapping developed by M. A. Lavrent'ev (the M. A. Lavrent'ev congruence theorems) have been applied richly in the papers of Mikhail Alekseev himself, his pupils, and followers in the theory of quasiconformal mapping, in problems of hydrodynamics with free boundaries, in filtration theory, in numerical methods of solving applied problems, and other branches of mathematics and mechanics. The survey of the results obtained here and references to appropriate papers can be found in [3, 4], for example. The translation from Ukrainian of a similarly titled paper of M. A. Lavrent'ev ([5] (1947)) is the first Russian publication of the complete proof of the classical theorem on the existence of a solitary wave announced by the author in 1943 in Doklady Akademii Nauk SSSR [6]. This article was translated by M. P. Shcherbyak and edited by V. N. Monakhov.

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Let $z_1 = x_1 + iy_1$ be an arbitrary point on the line Γ. Let us draw a circle C tangent to Γ through z_1 and a normal T to Γ to intersect the line Γ_0 at the point z_0, for instance; let us draw a circle C_0 tangent to Γ_0 through the point z_0. The domain bounded by C_0 and C will be denoted by Δ_{z_1}. Let us map conformally

$$\zeta = f(z, z_1)$$

the domain Δ_{z_1} onto the strip $0 < \eta < h$ under the condition that the vertices of the crescent Δ_{z_1} would go over into the points $\pm\infty$.

We obtain

$$|f'(z_1, z)| = \frac{h}{y - y_0}\left[1 - \frac{1}{3}(y - y_0) y'' + \frac{1}{6}(y - y_0) y_0 + \frac{11}{6} y'^2 - \frac{5}{3} y_0' y' + \frac{1}{3} y_0'^2\right] + r, \tag{2}$$

where

$$r = r\left(y_0, y, y_0', y', y_0'', y''\right)$$

is a function such that the expansion of the function $r(\pm y_0, ty, ty_0', \ldots)$ in t starts with the third power of t.

For sufficient smoothness of the lines Γ_0, Γ the quantity $|f'(r_1, z)|$ will yield an approximate value for $|f'(z_1, \Gamma_0, \Gamma)|$. Let us find the estimate and properties of the remainder term of this approximation. Henceforth, for simplicity in the writing, let us agree k and θ denote quentities which remain bounded as the appropriate parameters, particularly h, tend to zero. Let $\varphi_0(x)$ and $\varphi(x)$ denote, respectively, the differences between ordinates of the points C_0, Γ_0 and C, Γ; where this difference is not defined, we consider the functions φ_0 and φ to be given so that they remain continuous. Let us define the lines $\Gamma_0(\tau)$ and $\Gamma(\tau)$ by the equations

$$y = y_0(x) + \tau\varphi_0(x) = \varphi_0(x, \tau);$$
$$y = y(x) - \tau\varphi(x) = \varphi(x, \tau).$$

Letting $V(\tau)$ denote the absolute value of the derivative of the function $\zeta = f[z, \Gamma_0(\tau), \Gamma(\tau)]$ at the point z_1, which realizes the conformal mapping of the domain $D[\Gamma_0(\tau), \Gamma(\tau)]$ into the strip $0 < \eta < h$ under the condition of correspondence of the infinitely remote points, we will have [2]

$$\frac{1}{v(\tau)} \frac{d}{d(\tau)} V(\tau) = \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{|f'[z, \Gamma_0(\tau), \Gamma(\tau)]|\, \varphi(x) \cos\arg f' dt}{\operatorname{sh}^2 \frac{\pi}{2} \frac{\xi_1 - t}{h}} + \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{|f'[z, \Gamma_0(\tau), \Gamma(\tau)]|\, \varphi(x) \cos\arg f' dt}{\operatorname{ch}^2 \frac{\pi}{2} \frac{\xi - t}{h}}, \tag{3}$$

where z is a point of $\Gamma(\tau)$ which corresponds to a point t of the upper boundary of the strip under the mapping noted, and ξ_1 corresponds to z_1. Hence, setting

$$\bar{\varphi}(t) = \int_0^1 |f'(z, \Gamma_0(\tau), \Gamma(\tau)|\, \varphi(x) \cos\arg f' dt, \quad z \in \Gamma(\tau),$$

$$\bar{\varphi}_0(t) = \int_0^1 |f'(z, \Gamma_0(\tau), \Gamma(\tau)|\, \varphi_0(x) \cos\arg f' dt, \quad z \in \Gamma_0(\tau),$$

we have

$$\log \frac{V(1)}{V(0)} = \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{\bar{\varphi}(t)\, dt}{\operatorname{sh}^2 \frac{\pi}{2} \frac{\xi_1 - t}{h}} + \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{\bar{\varphi}_0(t)\, dt}{\operatorname{ch}^2 \frac{\pi}{2} \frac{\xi_1 - t}{h}},$$

from which we obtain the following form for the remainder term R:

$$R = \log |f'(z_1, \Gamma_0, \Gamma)| - \log |f'(z_1, z_2)| = \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{k\varphi_0(\theta - t)\, dt}{\mathrm{sh}^2 \frac{\pi}{2} \frac{\xi - t}{h}} - \frac{\pi}{4} \int \frac{k\varphi_0(\theta - t)\, dt}{\mathrm{ch}^2 \frac{\pi}{2} \frac{\xi - t}{h}} \tag{4}$$

or noting that

$$\varphi(x) = y(x) - y(x_1) - y'(x_1)(x - x_1) - \frac{1}{2} y''(x_1)(x - x_1)^2 - \theta |y''|^3 (x' - x_1)^3,$$

$$\varphi_0(x) = y_0(x) - y\left(x_1^{(0)}\right) - y_0'\left(x_1^{(0)}\right)(x - x_1) - \frac{1}{2} y_0''\left(x_1^{(0)}\right)(x - x_1)^2 + \theta_0 |y''|^3 (x - x_1)^3,$$

we have

$$|R| < k\{\max |y_0'''| + \max |y'''|\} h^2. \tag{5}$$

3. WAVES ON AN ARBITRARY BOTTOM AND RAYLEIGH WAVES

Turning to steady wave motions, let us set the value

$$|f'(z, \Gamma_0, \Gamma)|^2 = \frac{(h - v)^2}{(y - y_0)^2} \left(1 + \frac{2}{3} yy'''\right), \tag{6}$$

in the first approximation in relationship (1); then we obtain

$$\left(\frac{h - v}{y - y_0}\right)^2 \left(1 + \frac{2}{3} yy'''\right) = C - \lambda y$$

or discarding higher-order infinitesimals

$$\left(\frac{h - v}{y}\right)^2 \left[1 + \frac{2}{3} yy'' + 2\frac{y_0}{y} + 3\left(\frac{y_0}{y}\right)^2\right] = C - \lambda y.$$

Hence

$$yy'' = -\frac{3}{2} - 3\frac{y_0}{y} - \frac{9}{2}\left(\frac{y_0}{y}\right)^2 - \frac{3}{2} \frac{C}{(h - v)^2} y^2 - \frac{3}{2} \frac{\lambda}{(\lambda - v)^2} y^3. \tag{6'}$$

Let us first consider the case when $y_0 = 0$, $v = 0$. In this case, (6) becomes

$$yy'' = -\frac{3}{2} - \frac{3}{2} \frac{C}{h^2} y^2 - \frac{3}{2} \frac{\lambda}{h^2} y^3 = \varphi(y). \tag{6''}$$

Let us select the values of the constants so that the maximum of $\varphi(y)$ is reached at the point $y = h$ and the spacing between the positive roots of $\varphi(y)$ would be on the order of h^2; then

$$\varphi'(y) = \frac{3C}{h^2} y - \frac{9}{2} \frac{\lambda}{h^2} y^2.$$

Hence, setting $y = h$, we obtain

$$2C - 3\lambda h = 0. \tag{7}$$

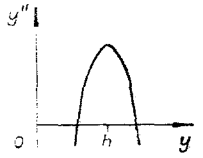

Fig. 1

Now if it is assumed that one of the roots of $\varphi(y)$ equals $h^2 + h$, then we obtain another relationship to determine C and λ:

$$-1 + \frac{C}{h^2}(h + h^2)^2 - \frac{\lambda}{h^2}(h + h^2)^3 = 0. \tag{7'}$$

From (7) and (7') we obtain

$$\lambda = \frac{2}{h} \frac{1}{1 - 3h^2 + 2h^3}$$

or retaining the principal terms, we further obtain

$$\begin{aligned} \lambda &= \frac{2}{h} + 6h; \\ C &= 3 + 9h^2. \end{aligned} \tag{8}$$

Let us note some properties of the function $\psi = (1/y)\,\varphi(y)$ for the values of λ and C taken.

The zeros y_1 and y_2 of the function ψ are

$$y_{1,2} = h \pm h^2 + \theta h^3. \tag{9}$$

The function ψ is positive in the interval $y_1 < y < y_2$ and at the point

$$y_0 = h + \theta h^3$$

reaches a maximum equal to

$$\psi_{\max} = \psi(y_0) = \frac{9}{2} h + \theta h^2.$$

Outside the interval $y_1 \leq y \leq y_2$ the function ψ is negative for $y > 0$ and convex from the left; it is convex from the right for $y < 3h/2$. For $h/4 < y < 5h/4$.

$$\psi''(y) = \frac{k}{h^3},$$

and, moreover,

$$\psi'(h \pm \theta h^2) = \mp \frac{k}{h}.$$

It will henceforth be convenient to give (6') another form. To do this we set

$$y_1(x) = y(x) - y_0(x)$$

and replace the function $y_0(x)$ by the function

$$\eta_0(x) = y_0(x) - v.$$

The relationship (6) becomes

$$\left(\frac{h - v}{y_1}\right)^2 \left(1 + \frac{2}{3} y_1 y_1''\right) = C - \lambda(y_0 + v + \eta_0). \tag{10}$$

The equation obtained is equivalent to (6) to the accuracy of infinitesimals of order higher than h^2 because of the estimates for ψ' and ψ''. Substituting their expression (8) in place of C and λ in (10), and replacing the right side by a quadratic term, we obtain

$$y'' = \frac{9}{2} h + \frac{1}{2} \frac{v^2}{h^3} - \frac{3\eta_0}{h^2} \frac{9}{2h^3} \left[y - \left(h + \frac{1}{3} v\right)\right]^2 = \psi(y, v). \tag{11}$$

For $\eta = v = 0$ we call the integral curve of (11) a Rayleigh wave [1].

Setting $\eta = v = 0$, we obtain

$$y'' = \frac{9}{2} h - \frac{9}{2h^3}(y-h)^2. \tag{11'}$$

The first integral will be

$$\left(\frac{dy}{dx}\right)^2 = 9hy - \frac{3}{h^3}(y-h)^3 + 9A, \quad A = \text{const}. \tag{12}$$

Let us study the change in the integrals of (11') which reach a maximum at $x = 0$ depending on the initial ordinate $y(0) = h + \alpha$, $\alpha > 0$.

The maximum condition yields

$$h^2 + h\alpha - \frac{\alpha^3}{3h^3} + A = 0,$$

from which

$$A = \frac{\alpha^3}{3h^3} h^2 - h\alpha. \tag{12'}$$

Let us find the value y, $y \neq h + \alpha$ for which $dy/dx = 0$. We have

$$hy - \frac{1}{3h^3}(y-h)^3 - h\alpha - h^2 + \frac{\alpha^3}{3h^3} = 0,$$

which yields after dividing by $y - h - \alpha$

$$(y-h)^2 + \alpha(y-h) + (\alpha^2 - 3h^4) = 0$$

or

$$y = h + \alpha, \quad (\alpha) = h - \frac{\alpha}{2} + \frac{\sqrt{3}}{2}\sqrt{4h^4 - \alpha^2}. \tag{13}$$

Hence, we see that the greatest value of α, for which we obtain a wave equals $2h^2$. Moreover, it is evident that for α infinitely close to h^2 the integral curve will be close to the line $y = h + h^2$, which means that the plus sign should be taken in (13).

From the calculations presented and directly from the form of (11') we have the result that for each value of α

$$-h^2 \leqslant \alpha < 2h^2$$

there exists an integral curve $y = Y(x, \alpha)$ of (11') which has a maximum for $x = 0$ and has the finite period $2\omega(\alpha)$:

$$Y(x + 2\omega(\alpha), \alpha) = Y(x, \alpha),$$

and, furthermore, it is evident that

$$Y(-x, \alpha) = Y(x, \alpha).$$

The period 2ω is determined by integrating (12),

$$\omega = \frac{h^{3/2}}{\sqrt{3}} \int_{h+\alpha_1(\alpha)}^{h+\alpha} \frac{dz}{\sqrt{3h^4(z-\alpha) - (z^3 - \alpha^3)}}, \tag{14}$$

where

$$\alpha_1(\alpha) = -\frac{\alpha}{2} - \frac{\sqrt{3}}{2}\sqrt{4h^4 - \alpha^2}.$$

It is seen from the expression obtained for ω that for $\alpha \to h^2$ the half-period ω tends to $\pi\sqrt{h}/3$

$$\lim_{\alpha \to h^2} \omega(\alpha) = \frac{\pi\sqrt{h}}{3},$$

where ω increases as α rises and ω tends to ∞ as $\alpha \to 2h^2$, the integral curve will achieve a single maximum for $\alpha = 2h^2$ and the line $y = h - h^2$ will be the asymptote of this curve. The curve $y = Y(x, 2h^2)$ yields an approximate profile of a "solitary wave."

Let us find the asymptotic expression for $\omega(\alpha)$ for values of α close to $2h^2$. To do this, let us represent ω as

$$\omega(\alpha) = \frac{h^{\frac{3}{2}}}{\sqrt{3}} \int_{\alpha_1}^{\alpha} \frac{dz}{\sqrt{(\alpha - z)(z - \alpha_1)(z - \alpha_2)}},$$

where $\alpha_2 = \alpha_2(\alpha)$ is determined from (13) and the minus sign must be taken in front of the radical.

Let us set

$$\alpha - \alpha_1 = \frac{3}{2}\alpha - \frac{\sqrt{3}}{2}\sqrt{4h^4 - \alpha^2} = 2\delta,$$

$$\alpha_1 - \alpha_2 = \sqrt{3}\sqrt{4h^4 - \alpha^2} = \varepsilon\delta, z = \delta t;$$

then

$$\omega(\alpha) = \frac{h^{3/2}\delta^{-1/2}}{\sqrt{3}} \int_{-1}^{1} \frac{dt}{\sqrt{(1 - t^2)(1 + \varepsilon - t)}} = \frac{h^{3/2}\delta^{-1/2}}{\sqrt{6}}\left[\log\frac{2}{\varepsilon} + A + B\varepsilon\log\frac{1}{3}\right]. \tag{15}$$

Instead of the variable α, let us introduce the variable τ,

$$\alpha = 2h^2 - \tau h^2;$$

then we will have for small τ

$$\delta = \frac{3}{2}h^2\left(1 - \frac{\sqrt{3}}{3}\sqrt{\tau} + \ldots\right); \ \varepsilon = \frac{4\sqrt{3}}{3}\sqrt{\tau} + \ldots$$

Finally,

$$\omega(\alpha) = \frac{1}{6}h^{\frac{1}{2}}\log\frac{1}{\tau} + \frac{\sqrt{3}}{36}h^{\frac{1}{2}}\tau^{\frac{1}{2}}\log\frac{1}{\tau} + \ldots$$

Starting from the relationships (11'), (12), and (12'), the following estimates can be obtained for the slope and curvature of the Rayleigh wave for any values of $|x| \leq \omega(\alpha)$:

$$Y'(x, \alpha) < kh^{\frac{3}{2}}e^{-\frac{3x}{\sqrt{h}}},$$
$$Y''(x, \alpha) < khe^{-\frac{3x}{\sqrt{h}}}, \tag{16}$$
$$Y'''(x, \alpha) < h^{\frac{1}{2}}e^{-\frac{3x}{\sqrt{h}}}.$$

The first of the estimates justifies the fact that the term containing y' has been discarded in going from (2) to (6), since this term is on the order of h^3 in conformity with (16).

Let us turn to the general equation (11) and let us establish the following property of its integrals. Let $y = z(x)$ be an integral of (11), $Z(0) = h + \alpha$, $Z(-x) = Z(x)$, $Z(x + 2\omega) = Z(x)$ for $v = 0$ and for $\eta = \eta_0(x)$, $\eta_0(-x) = \eta_0(x)$, $\eta_0(x + 2\omega) = \eta_0(x)$ $|\eta_0(x)| < \beta h^3$, and let the function $\eta = \eta(x)$ be such that the integral of (11) agrees with $Z(x)$. Let $Z(x)$ denote an integral of (11) for $\eta = \eta(x) - \Delta$, where Δ = const so that $Z_1(0) = h + \alpha$, $Z'(0) = 0.1$.

LEMMA 1. If $2h^2 - \alpha$ and β are sufficiently small, then for $x < \omega(\alpha)$ the difference

$$\delta Z = Z_1(x) - Z(x)$$

has the very same sign as Δ and $|\delta Z|$ grow, where $x = k\sqrt{h}$

$$|\delta Z| > \frac{k\Delta}{h}.$$

Proof. Because of the continuity of the integral as a function of the parameters of the equation, it is sufficient to examine the case when $\alpha = 2h^2$ and $\eta_0(x) = 0$. Hence, let Z go over into $Y(x, \alpha)$ and Z_1 into $Y_1(x_1, \alpha)$. Noting this, let us first consider a particular case of (11):

$$y'' = \frac{9}{2}h + \frac{3\bar{\Delta}}{h^2} - \frac{9}{2h^3}(y-h)^2. \qquad (11'')$$

Let us hence assume that $\bar{\Delta} = 0$ for $2h^2 > y \geq y_0$ and $\bar{\Delta} = \Delta$ for lesser values of y. Let us consider the integral $y = y(x)$ of this equation under the initial data

$$y(x_0) = Y(x_0, 2h^2),$$
$$y'(x_0) = Y'(x_0, 2h^2),$$

where x_0 is determined from the equation $Y(x_0, 2h^2) = y_0$.

Let

$$x = x(y),$$
$$\bar{x} = \bar{x}(y)$$

be functions inverse to the functions $y = Y(x, 2h)$ and $y = y(x)$, respectively. In conformity with (11'), (11"), and the initial conditions, we obtain

$$x = \frac{h^{\frac{3}{2}}}{\sqrt{3}}\int_y^{y_0}\frac{dy}{(y-h+h^2)\sqrt{h+2h^2-y}} = \int_y^{y_0}\frac{dy}{\sqrt{v}},$$

$$\bar{x} = \int_y^{y_0}\frac{dy}{\sqrt{U+\frac{27\Delta}{h^2}(y-y_0)}} = \int_y^{y_0}\frac{dy}{\sqrt{U}} + \frac{27\Delta}{2h^2}\int_y^{y_0}\frac{y_0-y}{U^{\frac{3}{2}}}dy.$$

Therefore,

$$\delta x = \bar{x} - x = kh^{\frac{5}{2}}\Delta\int_{y-h}^{y_0-h}\frac{y_0-h-y}{(y+h^2)^3(2h^2-y)^{\frac{3}{2}}}dy.$$

Noting that

$$y(x) - Y(x, 2h^2) \equiv Y'(x, 2h^2)\delta x,$$

and introducing the new variable u,

$$y = h + 2h^2 - h^2u^2,$$
$$y_0 = h + 2h^2 - h^2u_0,$$

we find

$$\delta y = y(x) - Y(x, 2h^2) = \frac{k\Delta}{h}(3 - u^2)u^2 \int\limits_{u_0}^{u} \frac{u^2 - u_0^2}{(3 - u^2)^3 u^2} du,$$

where k is a numerical constant, $0 < u_0 < u < \sqrt{3}$. Differentiating the last relation with respect to u, we find

$$\frac{d}{du}\delta y = \frac{k\Delta}{h}\left\{\frac{u^2 - u_1^2}{(3 - u^2)^2} - 2(2u^2 - 3)u \int\limits_{u_0}^{u} \frac{u^2 - u_0^2}{(3 - u^2)^2 u^2} du\right\}.$$

The right side of this latter equation is explicitly positive for $u^2 < 3/2$; it can be seen by direct substitution that this holds for all u_0 and u, $0 < u_0 < u < \sqrt{3}$. This means that the variation δy increases. Moreover, calculations carried out show that for $x_0 = 0$ and values of x on the order of $\sqrt{h}$ the variation δy is on the order of Δ/h.

Forming a variational equation from (11), we obtain the following differential equation for δy:

$$\delta y'' = -\frac{9}{h^3}(Y - h)\delta y + \frac{3\bar{\Delta}}{h^2},$$

$$\bar{\Delta} = 0 \text{ for } x < x_0 \quad \text{and} \quad \bar{\Delta} = \Delta \text{ for } x > x_0.$$

From the monotonicity of δy proved above for any x_0 we arrive at the following: If $\varphi(x)$ is a nondecreasing positive function, then $\varphi'(x) \geq 0$, $\varphi(x) > 0$, $x > 0$, and if z (x)

$$z(0) = z'(0) = 0$$

is an integral of the equation

$$z'' = -\frac{9}{h^3}(Y - h)z + \varphi(x),$$

then z (x) is a nondecreasing function.

Let us turn to (11) and let us form its variational equation when the function η is the increment $-\Delta$; since (11) is Y under the condition that $\Delta = 0$, we then obtain for δY

$$\delta Y'' = -\frac{9}{h^3}\left(Y - h - \frac{1}{3}v\right)\delta Y + \frac{3\Delta}{h^2}.$$

Let us equate the integral of this equation to the integral of the equation

$$\delta Y_1'' = -\frac{9}{h^3}(Y - h)\delta Y_1 + \frac{3\Delta}{h^2},$$

$$\delta Y(0) = \delta Y_1(0) = \delta Y'(0) = \delta Y_1'(0) = 0:$$

by setting

$$X = \delta Y - \delta Y_1,$$

we obtain

$$X'' = -\frac{9}{h^3}(Y - h)X + \frac{1}{3}v\delta Y.$$

For infinitesimal v we can replace δY by the nondecreasing function δY_1 and in conformity with the above, we will have

$$X' \geq 0;$$

however, reasoning as before, let us prove the monotonicity of the integral of the equation

$$X'' = -\frac{9}{h^3}(Y - h - dh)X + \varphi(x),$$
$$\varphi(x) > 0, \ \varphi'(x) \geqslant 0, \ dh > 0.$$

Hence, by induction we obtain that for any $v > 0$ we have $X' > 0$, which proves the lemma completely.

5. THE OPERATOR I AND ITS VARIATION

Let us introduce the following differential operator:

$$I(\Gamma_0, \Gamma) = \left(\frac{h - v}{y}\right)^2 \left(1 + \frac{2}{3} yy''\right) - C + \lambda(y + v + \eta). \tag{17}$$

Let us study the variation of this operator upon making the transition from the line Γ to the nearest line $\overline{\Gamma}$.

LEMMA 2. Let $f(x)$ be a continuous function $|f(x)| < vh^2$, $v \to 0$ as $h \to 0$, and along the line Γ: $y = y(x)$, $y'(0) = 0$, $y(0) = h + \alpha$

$$I(\Gamma_0, \Gamma) = f(x), \tag{17'}$$

while along the line $\overline{\Gamma}$: $y = \bar{y}(x)$, $\bar{y}'(0) = 0$, $\bar{y}(0) = y(0)$

$$I(\Gamma_0, \overline{\Gamma}) = \bar{f}(x),$$

where

$$|\bar{f}(x) - f(x)| \leqslant \varepsilon.$$

Then for $0 \leq v < kh^2$, $0 \leq \eta < kh^2$ we have

$$|\bar{y}(x) - y(x)| \leqslant \varepsilon \operatorname{ch} \frac{(3 + \delta)x}{\sqrt{h}}, \tag{18}$$

where $\delta \to 0$ as $h \to 0$.

Proof. Indeed, taking account of (11) we can represent (17') as

$$y'' = \psi(y, v) + \mu(y) + \frac{3}{2} \frac{y}{(h - v)^2} f(x),$$

where $\mu(y)$ is a continuous and differentiable function of y and $\mu(y) = 0(\psi)$ and $\mu'(y) = 0(\psi')$, respectively, in the neighborhoods of the zeros of ψ and ψ'.

Setting

$$\bar{f}(x) - f(x) = \varepsilon(x), \ |\varepsilon(x)| < \varepsilon, \ \bar{y}(x) - y(x) = \varphi(x)$$

and forming the variational equation for (18), we obtain

$$\varphi'' = \left\{\psi' + \mu' + \frac{3}{2} \frac{f}{(h - v)^2}\right\} \varphi + \frac{k\varepsilon(x)}{h}$$

or taking into account the expression for ψ and the conditions for v and η

$$\varphi'' = \frac{\theta}{h} \varphi + \frac{k\varepsilon(x)}{h}, \tag{19}$$

where θ satisfies the inequality

$$-G - 0(h) \leqslant \theta \leqslant G + 0(h).$$

Noting that $\varphi(0) = 0$ and $\varphi'(0) = 0$ for our case and that we obtain the greatest value for φ'' by putting $\theta = G + 0(G)$ and $\varepsilon(x) = \varepsilon$ in (9), we find the original estimate by integration.

6. AUXILIARY PROBLEM

Retaining the notation used in Sec. 1, let us examine the following problem: Let Γ_0: $y = y_0(x)$ be such that $y_0(x)$ is periodic with period 2ω, admits of a unique maximum at $x = 0$ for $|x| \leq \omega$, $y_0(-x) = y_0(x)$, and, moreover,

$$|y_0(x) - v| < kh^3, \quad |y_0'(x)| < kh^{2+\nu}, \quad |y_0''(x)| < kh^{1+\nu}, \qquad (20)$$
$$v < kh^2, \ \nu > 0.$$

Let Γ_0' denote the line $y = y_0(x) + \Delta$, where Δ is some constant. Determine the line $\Gamma: y = y(x)$

$$y(x+2\omega) = y(x), \ y(0) = h + \alpha$$

such that $y(\Gamma_0', \Gamma) = 0$, where α is a given number and the number Δ must be determined, $\Delta = \Delta\{y_0\}$. Let

$$y(x) = H\{y_0(x)\} = H\{y_0\}$$

denote the solution of the problem posed.

Let us henceforth limit ourselves to the case when the numbers C and λ are determined from (8) but the number α belongs to the range $h^2 < \alpha < 2h^2 + 0(h^2)$. Moreover, let us consider the solution of the problem posed so that H would be even and would admit a single maximum in the interval of the period at $x = 0$.

Assuming the solution of the problem posed exists and $|\Delta\{y_0\}| < kh^3$, let us establish a number of its properties.

7. ESTIMATES OF DERIVATIVES OF THE WAVE LINE

Let $y = y_0(x)$ be the solution of the posed auxiliary problem. Let us find estimates for y', y'', and y'''. Without limiting the generality, we can additionally consider that $\Delta\{y_0\} = 0$ under the condition $|\Delta\{y_0\}| < kh^3$ because of (20).

<u>LEMMA 3.</u> We have

$$|y'(x)| < \frac{\bar{k}h}{\log \frac{1}{h}},$$

where $\bar{k}$ is a constant dependent only on the constant k introduced above.

<u>Proof.</u> Let us note that because of the elementary variational lemma from conformal mapping theory [2] we have

$$|y(x) - h| > kh^2.$$

Let c denote the maximum value of $|y'(x)|$, and let

$$y'(x_0) = c.$$

In order to obtain the desired estimate for c let us consider the derivative of the function

$$P = \log V = \log |f'(z_0, \Gamma_0', \Gamma)|$$

with respect to x at the point $z_0 = x_0 + iy(x_0)$. Because of (1), at the point under consideration, we have

$$\frac{dP}{dx} = \frac{\lambda c}{2V^2} = -\left(\frac{1}{h} + 3h\right)\frac{c}{V^2} = -\frac{c}{h} + \theta c. \tag{21}$$

Now, let us find the upper bound of this quantity by starting from the geometric conditions imposed on Γ_0 and Γ. To this end, let us construct a domain A bounded by: 1) a segment α tangent to Γ at the point $z_0 = x_0 + iy(x_0)$ enclosed between the lines $y = h + kh^2$ and $y = h - kh^2$, 2) rays of the lines $y = h \pm kh^2$ that issue from the ends of the segment constructed above; 3) the segment α_0 of the line $y = kh^{2+\nu}(x - x_1) + v$ (where x_1 is the abscissa of the middle of the segment α) enclosed between the lines $y = v \pm kh^3$; 4) rays of the lines $y = v \pm kh^3$ which issue from the ends of the segment α_0. Let $\zeta = f_1(z)$, $f_1(\pm\infty) = \pm\infty$ map A onto the strip $v < \eta < h$ and $P_1 = \log|f_1'(z)|$.

Because of the above-mentioned lemma, it can be seen that

$$\left[\frac{dP_1}{d\xi}\right]_{x=x_1} > \left[\frac{dP}{d\xi}\right]_{x=x_0} = \frac{dP}{dx}(1 + \theta h). \tag{21'}$$

For the mapping $\zeta = f_1(z)$ let the segment $(-\xi_1, \xi_1)$ correspond to the segment d and the segment $(\xi_1^{(0)}, \xi_2^{(0)})$ to the segment d_0; then, evidently,

$$\xi_1 = \theta\frac{h^2}{c}, \quad \xi_1^{(0)} = -\theta_1 h^{1-\nu}, \quad \xi_2^{(0)} = \theta_2 h^{1-\nu}.$$

Let us construct a harmonic function Q in the strip $v < \eta < h$, which equals c on the segment $(-\xi_1, \xi_2)$ of the line $\eta = h$, $kh^{2+\nu}$ on the segment $(\xi_1^{(0)}, \xi_2^{(0)})$, and zero on the rest of the boundary.

We have

$$\left|\frac{dP_1}{d\xi}\right|_{x=x_1} = \left|\frac{\partial Q}{\partial\eta}\right|_{\substack{\xi=0\\ \eta=h}} < -\frac{c}{h} + \frac{kh^{1+\nu}}{h^2}\int\limits_{-kh}^{kh}\frac{dt}{\operatorname{sh}^2\frac{\pi}{2}\frac{t}{h-v} + k} + \frac{\theta c}{h^2}\int\limits_{\frac{kh^2}{c}}^{\infty}\frac{dt}{\operatorname{sh}^2\frac{\pi}{2}\frac{t}{h-v}} = -\frac{c}{h} + kh^{1+\nu} + \frac{\theta c}{h}\int\limits_{\frac{kh}{c}}^{\infty}\frac{dt}{\operatorname{sh}^2\frac{\pi}{2}t}. \tag{21''}$$

Hence, comparing (21), (21'), and (21''), we obtain

$$\int\limits_{\frac{kh}{c}}\frac{dt}{\operatorname{sh}^2 t} < \theta h.$$

which means

$$\frac{kh}{c} > \theta\log\frac{1}{h},$$

which is the final estimate for c.

LEMMA 4. We have $|y''(x)| < kh$.

Proof. Let us perform a calculation in the variables ξ, η and in conformity with this, let us set

$$\overline{y}(\xi) = y[x(\xi, h)], \quad \overline{y}_0(\xi) = y[x_0(\xi, v)].$$

for points of the wave line and the bottom, where $x = x(\xi, \eta)$ is the real part of the function inverse to $f(z, \Gamma_0', \Gamma)$. Moreover, let us introduce the conjugate harmonic functions $P(\xi, \eta)$, $Q(\xi, \eta)$:

$$P = -\log U = -\log|f'(z, \Gamma_0', \Gamma)|.$$
$$Q = -\arg f'(z, \Gamma_0', \Gamma).$$

At points of the wave line we have

$$y' = \operatorname{tg} Q,$$
$$y'' = \frac{1}{\sqrt{\cos^2 Q}} \frac{\partial Q}{\partial \xi} = (1 + \theta h) Q'. \tag{22}$$

Analogously, along the bottom

$$y_0^{\bullet} = (1 + \theta h) Q'. \tag{22'}$$

The last relationships show that Q' has the very same order of magnitude as y''. Let ξ_1 be a point at which |Q'| reaches the absolute maximum; for definiteness, let us assume that

$$Q'(\xi) = \frac{\partial Q(\xi, h)}{\partial \xi} \geqslant Q'(\xi_1). \tag{23}$$

Let us calculate the value of $\partial^2 P/\partial \xi^2 = P''$ at the point (ξ_1, h). In conformity with (1), we have

$$P = -\frac{1}{2}\log(C - \lambda y)$$

or replacing C and λ by their values from (9) and assuming

$$y = h + \tau h, \quad |\tau| < kh,$$

we obtain

$$P = \tau + kh^2 + \theta\tau^2, \tag{24}$$

from which

$$P' = \frac{v}{h} y', \tag{25}$$

$$P'' = \frac{v^2}{h} y'' - \frac{2}{h^2} V y'^2 = (1 + \theta h)\frac{Q_1'}{h} + \frac{k}{\log^2 \frac{1}{h}}.$$

We can express this same quantity in terms of the value of Q' by means of the Poisson formula. Assuming $h - v = h_1$, we obtain

$$P'' = \frac{\partial^2 Q}{\partial \xi \partial \eta} = \frac{Q_1'}{h}(1 + \theta h) + \frac{\pi}{4}\frac{1}{h_1^2}\int_{-\infty}^{\infty} \frac{Q' - Q_1'}{\operatorname{sh}^2 \frac{\pi}{2}\frac{\xi - \xi_1}{h_1}}\, d\xi + \frac{k}{h_1^2}\int_{-\infty}^{\infty} \frac{Q(\xi, v)\, d\xi}{\operatorname{sh}^2 \frac{\pi}{2}\frac{\xi - \xi_1}{h_1}} + k. \tag{26}$$

Equating (25) and (26), let us show that $Q' \to 0$ as $h \to 0$. Indeed, if (22') and (20) are taken into account, then

$$\frac{1}{h_1^2}\int_{-\infty}^{\infty} \frac{Q' - Q_1'}{\operatorname{sh}^2 \frac{\pi}{2}\frac{\xi - \xi_1}{h_1}}\, d\xi < \theta Q_1' + 0(1)h, \tag{27}$$

is obtained from these equations.

Moreover, by virtue of Lemma 2, $Q \to 0$ as $h \to 0$. Hence, setting $|Q| = \varepsilon$, it can be seen that

$$\frac{1}{h_1^2}\int_{-\infty}^{\infty} \frac{Q' - Q_1'}{\operatorname{sh}^2 \frac{\pi}{2}\frac{\xi - \xi_1}{h_0}}\, d\xi > \frac{2Q_1'}{h_1^2}\int_{\frac{\varepsilon h}{Q_1'}}^{\infty} \frac{d\xi}{\operatorname{sh}^2 \frac{\pi}{2}\frac{\xi}{h_1}} = \frac{4Q_1'}{\pi h}\left(\operatorname{cth}\frac{\pi\varepsilon}{2Q_1'} - 1\right). \tag{27'}$$

This means that starting from the opposite $Q_1' > K$,

$$\operatorname{cth}\frac{2\varepsilon}{2Q_1'} - 1 < kh$$

or

$$Q_1' < \frac{\theta\varepsilon}{\log\frac{1}{h}}, \tag{28}$$

which contradicts the assumption.

To obtain the desired estimate, let us use an expression for P in terms of the function $\bar{y}(\xi)$. In conformity with the Poisson formula, we have for the strip $0 < \eta < h_1 = h - v$

$$\frac{1}{V} = \frac{\partial x}{\partial \xi}\frac{1}{\cos Q} = \frac{\partial y}{\partial \eta}\frac{1}{\cos Q} = \frac{\bar{y} - \bar{y}_0}{h_1}\frac{1}{\cos\theta} + \frac{\pi}{4}\frac{1}{h^2\cos Q}\int_{-\infty}^{\infty}\frac{\bar{y}(t) - \bar{y}(\xi_0) - (t-\xi_0)\bar{y}'(\xi_0)}{\operatorname{sh}^2\frac{\pi}{2}\frac{t-\xi_0}{h_1}}\,dt + kh^2.$$

Or by virtue of Lemma 2, setting $\bar{y} - \bar{y}_0 = h_1 + \tau h_1$,

$$\frac{1}{V} = 1 + \tau + \frac{\pi}{4h_1^2}\int_{-\infty}^{\infty}\frac{\delta(t)\,dt}{\operatorname{sh}^2\frac{\pi}{2}\frac{t-\xi_0}{h_1}} + k\bar{y}(\xi_0)\bar{y}''(\xi_0) + \theta h^2, \tag{29}$$

where

$$\beta(t) = \bar{y}(t) - \bar{y}(\xi_0) - (t-\xi_0)\bar{y}'(\xi_0) - \tfrac{1}{2}(t-\xi_0)^2\bar{y}''(\xi_0).$$

By virtue of (24) we have

$$\frac{1}{V} = 1 + \tau + \theta h^2. \tag{30}$$

Now, let us assume from the opposite that $(1/h)\bar{y}''(\xi_0)$ can be arbitrarily large; then from (29) and (30) we obtain

$$\frac{1}{h^2}\int_{-\infty}^{\infty}\frac{\delta(t)\,dt}{\operatorname{sh}^2\frac{\pi}{2}\frac{t-\xi_0}{h}} \geqslant Kh\bar{y}''(\xi_0). \tag{31}$$

Under the same assumption, from (27) and (22) it is obtained that

$$\frac{1}{h^2}\int_{-\infty}^{\infty}\frac{Q - Q_1'}{\operatorname{sh}^2\frac{\pi}{2}\frac{t-\xi_0}{h_1}}\,dt \leqslant KQ_1' + 0(1). \tag{32}$$

To prove the lemma there remains to show that (31), (32), and (23) are contradictions. Writing the function Q in place of δ in (31) for this, we have

$$\delta(t) = \int_0^t dt\int_0^t [\bar{y}''(t) - \bar{y}''(\xi_0)]\,dt = \int_0^t dt\int_0^t\left[\frac{1}{V}\frac{Q'}{\cos^2 Q} - \frac{1}{V_1}\frac{Q_1'}{\cos^2 Q_1}\right]dt = (1+\theta h)\int_0^t dt\int_0^t (Q' - Q_1')\,dt + \int_0^t dt\int_0^t \theta Q'(V - V_1)\,dt +$$
$$+ \int_0^t dt\int_0^t \theta Q'(Q - Q_1)\,dt.$$

Starting from the assumption of Lemma 2 and the inequality (28), it can be shown that the last two members in the expression of $\delta(t)$ represented by (31) will yield quantities of the order of $0(Q_1')$. Noting, in addition, that the orders of smallness of $\bar{y}''(\xi_0)$ and Q_1' are the same and setting $Q' - Q_1' = \varphi(t+\xi_0)$, $\psi(t) = \int_0^t dt \int_0^t \varphi(t)\, dt$, we obtain the following inequalities:

$$\frac{1}{h^2}\int_0^\infty \frac{\psi(t)\,dt}{\operatorname{sh}^2\frac{t}{h}} \geqslant KhQ_1', \tag{33}$$

$$\frac{1}{h^2}\int_0^\infty \frac{\varphi(t)\,dt}{\operatorname{sh}^2\frac{t}{h}} = 0(1), \tag{34}$$

$$0 \leqslant \varphi(t) \leqslant 2Q_1'.$$

Integrating the left side of (34) twice by parts, we obtain

$$\frac{1}{h^2}\int_0^\infty \psi(t)\frac{d^2}{dt^2}\frac{1}{\operatorname{sh}^2\frac{t}{h}}\,dt = 0(1)$$

or

$$\frac{d^2}{dt^2}\frac{1}{\operatorname{sh}^2\frac{t}{h}} > \frac{k}{h^2}\frac{1}{\operatorname{sh}^2\frac{t}{h}}.$$

Therefore, we finally obtain $KhQ_1' < kh^2$ in (33), which yields the desired inequality.

LEMMA 5. Under the conditions of Lemma 4 we have

$$|y'(x)| < \bar{k}h^{3/2}.$$

Proof. Let $a = y'(x_0)$ be the maximum value of $y'(x)$. By virtue of Lemma 3 we have

$$y(x) - y(x_0) \geqslant a(x - x_0) - \tfrac{1}{2}\theta h(x - x_0)^2;$$

but by assumption $|y(x) - y(x_0)| < \theta_1 h^2$; therefore, for any x

$$a(x - x_0) - \tfrac{1}{2}\theta h(x - x_0)^2 < \theta_1 h^2;$$

in particular, for $x = x_0 + (a/\theta h)$

$$\frac{a^2}{\theta h} - \frac{1}{2}\frac{a^2}{\theta h} < \theta_1 h^2$$

or

$$a < \sqrt{2\theta\theta_1}\, h^{3/2}.$$

LEMMA 6. In addition, if $y_0'' < kh^2$, then

$$|y'''(x)| < \frac{k}{\log\frac{1}{h}}. \tag{35}$$

Proof. Retaining the method presented for the proof of Lemma 3, let us find the expression for the function $P''' = \partial^3 P/\partial\xi^3$, obtained from (24), on the one hand, and from the value of $Q'' = \partial^2 Q/\partial\xi^2$, on the other.

In conformity with (25) and Lemmas 3 and 4, we have

$$P''' = \frac{V^3}{h} y''' + \frac{2VV'}{h} y'' - \frac{4V}{h^2} y'y'' - \frac{2V'}{h^2} y'^2 = (1 + \theta h) Q'' + kh^{\frac{1}{2}}, \tag{36}$$

where it is evident that

$$y''' = (1 + \theta h) Q'' + kh^{\frac{3}{2}}.$$

Now, let us consider the point ξ_1 at which $|Q''|$ reaches the absolute maximum. For definiteness, let the function $Q''(\xi)$ also reach the absolute minimum at this point:

$$Q''(\xi) \geqslant Q''(\xi_1) = Q_1''.$$

In conformity with the Poisson formula, at the point $[\xi_1, \bar{y}(\xi_1)]$ we have

$$P''' = \frac{Q_1'''}{h_1} + \frac{\pi}{4} \frac{1}{h^2} \int\limits_{-\infty}^{\infty} \frac{Q'' - Q_1''}{\operatorname{sh}^2 \frac{\pi}{2} \frac{t - \xi_1}{h_1}} dt + \frac{k}{h^2} Q_1' \tag{37}$$

or since y_0'' and $Q'(\xi, v)$ in addition are of the order of h^2, the remainder term is bounded, and, therefore, from (36) and (37) we obtain

$$\frac{1}{h^2} \int\limits_{\xi_1}^{\infty} \frac{Q'' - Q_1''}{\operatorname{sh}^2 \frac{\pi}{2} \frac{t - \xi_1}{h_1}} dt > \theta Q_1'' + k,$$

where

$$0 \leqslant Q'' - Q_1'' < 2Q'',$$

and by virtue of Lemma 3

$$|Q'| < kh.$$

Hence,

$$\frac{1}{h_1^2} \int\limits_{\xi_1}^{\infty} \frac{Q'' - Q_1''}{\operatorname{sh}^2 \frac{\pi}{2} \frac{t - \xi_1}{h_1}} dt > \frac{2Q_1''}{h_1^2} \int\limits_{\frac{kh}{Q_1''}}^{\infty} \frac{dt}{\operatorname{sh}^2 \frac{\pi}{2} \frac{t}{h_1}}. \tag{38}$$

We therefore obtain a relationship analogous to (27'), which yields

$$Q_1'' < \frac{\theta}{\log \frac{1}{h}}.$$

8. ESTIMATE OF DERIVATIVES OF THE WAVE-LINE VARIATIONS

Let two lines $\Gamma_0 : y = y_0(x)$ and $\Gamma_0 : y = y_0(x) + \delta y_0(x)$ be given, where $y_0(x)$ satisfies the conditions of Sec. 7 and Lemma 6, and the function $\delta y_0(x)$ is such that

$$\begin{aligned} &|\delta y_0(x) - \varepsilon'| < \varepsilon_0, \\ &|\delta y_0'(x)| < kh^{5/2}, \\ &|\delta y_0''(x)| < kh^2. \end{aligned} \tag{39}$$

Moreover, the wave lines $\Gamma: y = y(x) = H\{y_0(x)\}$ and $\bar{\Gamma}: y = y(x) + \delta y(x)$ with the identical period 2ω correspond to the lines Γ_0 and $\bar{\Gamma}_0$, which satisfy the conditions taken in Sec. 7, and such that

$$|\delta y(x)| \leqslant \varepsilon, \quad \underset{\varepsilon_1 < \varepsilon}{\Delta} \{y_0(x)\} = 0, \quad |\Delta\{y_0 + \delta y_0\}| = 0. \tag{40}$$

Obtaining estimates for the three derivatives found for the function $\delta y(x)$ is the problem of Lemmas 3-6. Because we do not use the symmetry of $y(x)$ in the subsequent assumption, we can reduce the problem of estimating $|\delta y'|$, $|\delta y''|$, $|\delta y'''|$ to the estimation of the same quantities at some fixed neighborhood of the point $x = 0$ by translating the origin. By analogy with the previous section, let us perform the calculation in the variables ξ, η and in conformity with this, let us reduce the data of the condition to the variables ξ, η.

Let the correspondence

$$\xi = \xi(x), \ x = x(\xi); \ \xi(0) = x(0) = 0;$$
$$\xi = \xi_1(x), \ x = x_1(\xi); \ \xi_1(0) = x_1(0) = 0,$$

be established by virtue of the conformal mappings $z = z(\xi) = x(\xi, \eta) + iy(\xi, \eta)$, $z = z_1(\zeta) = x_1(\xi, \eta) + iy(\xi, \eta)$ of the domains $D(\Gamma_0, \Gamma) : D(\bar{\Gamma}_0, \bar{\Gamma})$ onto the strip $0 < \eta < h_1$ between points of the lines Γ, $\bar{\Gamma}$ and the line $\eta = h$ and the correspondence

$$\xi = \xi^{(0)}(x), \quad x = x^{(0)}(\xi);$$
$$\xi = \xi_1^{(0)}(x), \quad x = x_1^{(0)}(\xi),$$

between points of the lines Γ_0, $\bar{\Gamma}_0$ and the line $\eta = v$. We have

$$\xi_1(x) - \xi(x) = \int_0^x \left(\frac{V_1}{\cos Q_1} - \frac{V}{\cos Q} \right) dx, \tag{41}$$

where Q and Q_1 are the slopes of the curves Γ and $\bar{\Gamma}$ relative to the x axis and V and V_1 are the velocities of fluid motion at corresponding points of Γ and $\bar{\Gamma}$. Retaining only the principal terms in the right side of (41), we obtain

$$\xi_1(x) - \xi(x) = \int_0^x \frac{V_1 - V}{\cos Q} dx - \int_0^x \frac{V \sin Q}{\cos^2 Q} (Q_1 - Q) dx.$$

Noting that $|V_1 - V| < 2\varepsilon/h$ by virtue of (1), $|\theta| < kh^{3/2}$ by virtue of Lemma 4, and integrating the second integral by parts, we obtain

$$|\xi_1(x) - \xi(x)| < \frac{k\varepsilon}{h} x + kh^{3/2} \int_0^x (Q_1 - Q) dx < \frac{k\varepsilon}{h} x + kh^{3/2}\varepsilon, \tag{41'}$$

$$|\xi_1(x) - \xi(x)| < \frac{k\varepsilon}{h} x + kh^{3/2}_{\max} |Q_1 - Q| x. \tag{41''}$$

Moreover, $d\xi_1/dx = 1 + kh$. Hence,

$$|\delta x(\xi)| = |x_1(\xi) - x(\xi)| < \frac{k\varepsilon}{h} \xi + kh^{3/2}\varepsilon, \tag{42}$$

and also

$$|\delta x(\xi)| < \frac{k\varepsilon}{h} \xi + kh^{3/2}_{\max} |Q_1 - Q| \xi. \tag{42'}$$

It is impossible to obtain an analogous estimate for the correspondence between the line $\eta = 0$ and Γ_0; thus, for this case we estimate

$$\sigma(\xi) = \int_0^{\xi} |x_1(\xi) - x(\xi)|\, d\xi.$$

Let us first consider the case when $y_0(x) = v$ and $\varepsilon' = 0$. Under these conditions it can be shown by using Theorem 1' in [2] that we obtain the majorant for σ if we set

$$\delta y_0(x) = \varepsilon_0 \text{ for } 0 < x < \xi,$$
$$\delta y_0(x) = -\varepsilon_0 \text{ for } x < 0 \text{ and } x < \xi.$$

On the other hand, without changing the order of σ,

$$\delta y_0(x) = 0 \text{ for } |x| < \xi,$$
$$\delta y_0(x) = -\varepsilon \text{ for } x \geqslant \xi.$$

However, in this case we will have

$$\delta x'(t) = \frac{k\varepsilon}{h} + \frac{k\varepsilon_0}{h^2}\int_{\xi}^{\infty} \frac{d\tau}{\operatorname{sh}^3 \frac{t-\tau}{h_1}}, \quad t < \xi.$$

Integrating this equation twice, we finally obtain

$$\sigma(\xi) = \int_0^{\xi} |\delta x(\xi)\, d\xi < \frac{k\varepsilon_1}{h}\xi^2 + \varepsilon_0\xi, \quad \eta = 0.$$

It can be shown by an additional conformal mapping, that the same estimate will hold even in the general case.

For the lines Γ and $\bar{\Gamma}$ let us set

$$y = y[x(\xi)] = \bar{y}(\xi),$$
$$y = y[x_1(\xi)] + \delta y[x_1(\xi)] = \bar{y}(\xi) + \delta\bar{y}(\xi),$$

and analogously for the lines Γ_0 and $\bar{\Gamma}_0$,

$$y = y_0[x^{(0)}(\xi)] = \bar{y}_0(\xi),$$
$$y = y_0(x) + \delta y_0(x) = \bar{y}_0(\xi) + \delta\bar{y}_0(\xi).$$

Because of (41) and the condition $|y'| < kh^{1/2}$, we obtain

$$\delta\bar{y}(\xi) < \varepsilon + kh^{1/2}\varepsilon\xi.$$

Starting from (41'), let us estimate $\int_a^{\xi} |\delta\bar{y}_0|\, d\xi$, $a \geqslant 0$. We have

$$\int_a^{\xi} |\delta\bar{y}_0|\, d\xi \leqslant \int_a^{\xi} |\delta y_0|\, d\xi + \int_a^{\xi} |y_0'|\,|\delta x|\, d\xi \leqslant \varepsilon d\xi - a + |y_0'(\xi)|\,\sigma(\xi) + \int_a^{\xi} |x' y_0''|\,\sigma(\xi)\, d\xi < \varepsilon d\xi - a + kh^{3/2}\varepsilon\xi^2. \tag{43}$$

Let us note

$$|\delta\bar{y}'| < |\delta y'|\,(1 + \theta h) + kh\varepsilon\xi + kh^{1/2}\varepsilon. \tag{44}$$

Let us now find the estimate of $\delta y'$, $\delta y''$ and $\delta y'''$.

LEMMA 7. For $\varepsilon_0 < kh\varepsilon$ we have

$$|\delta y'| = |y_1'(x) - y'(x)| < \frac{\bar{k}\varepsilon}{h \log \frac{1}{h}},$$

where $\bar{k}$ is a constant dependent only on the constant k introduced earlier.

Proof. Let $m(\xi)$ denote a function equal to $2\varepsilon_1$ for $|\xi| < 1/k\sqrt{h}$ and $2k\sqrt{h}\varepsilon|\xi|$ for $|\xi| \geq 1/k\sqrt{h}$, and let $m_0|\xi|$ be a function equal to $2\varepsilon_0$ for $|\xi| < (\varepsilon_0/k\varepsilon)h^{-3/2}$, and $2kh^{3/2}\varepsilon\xi$ for the remaining values of ξ. By virtue of (42)

$$|\delta\bar{y}(\xi)| < m(\xi), \quad \int_0^{|\xi|} |\delta\bar{y}_0(\xi)|\, d\xi < m_0(\xi)\xi.$$

Let us examine the segment $|\xi| \leq 1/2x\sqrt{h}$. Let the function $|\delta\bar{y}'(\xi)|$ considered on this segment reach an absolute maximum at the point a_1, $|a_1| \leq 1/2k\sqrt{h}$, and for definiteness this will be the maximum for $\delta\bar{y}'(\xi)$. Under these conditions it can be seen that a point a exists on the segment $[a_1, 2\varepsilon/\delta\bar{y}'(a_1)]$ which possesses the following property: If a tangent L to the line $y = \delta\bar{y}(\xi)$ is drawn through the point $[a, \delta\bar{y}(a)]$, our line will be below L to the right of a and above L to the left; we will hence have at the point a

$$q = \delta\bar{y}'(a) \geqslant \delta\bar{y}'(a_1),\ \delta\bar{y}''(a) = 0. \tag{45}$$

To obtain the required estimate of q, let us express the value of the second derivative of the function

$$\delta x(\xi) = x_1(\xi) - x(\xi)$$

in terms of q. In conformity with (1), we have

$$\delta x'(\xi) = \frac{\cos Q_1}{V_1} - \frac{\cos Q}{V} = -\frac{\cos Q}{V^2}\delta V - \frac{\sin Q}{V}\delta\theta + \ldots,$$

where

$$V = \sqrt{c - \lambda y},\ V_1 = \sqrt{c - \lambda y_1},$$
$$\operatorname{tg} Q = \bar{y}'V,\quad \operatorname{tg} Q_1 = \bar{y}_1'V_1,$$

and by virtue of (8)

$$\delta V = -\frac{\delta y}{h}(1 + \theta h),$$
$$\delta Q = \cos^2 Q\left\{V\delta\bar{y}' + \bar{y}'\frac{\delta y}{h}(1 + \theta h)\right\},$$

which means

$$\delta x'(\xi) = \frac{\delta\bar{y}}{h} + \bar{y}'\frac{\delta\bar{y}}{h} + \theta h\frac{\delta\bar{y}}{h} + \bar{y}'\delta\bar{y}' + \ldots. \tag{46}$$

Hence, taking (45) into account, at the point a we will have

$$\delta x''(a) = \frac{\delta\bar{y}'(a)}{h}(1 + \theta h) = \frac{q}{h} + \theta h. \tag{47}$$

Now, let us find this same quantity by considering $x(\xi)$ as the boundary value of a function conjugate to the function $\delta y(\xi, \eta)$. We have

$$\delta x''(a) = \left[\frac{\partial^2 \delta y}{\partial \xi_0 \partial \eta}\right]_{\substack{\xi = a \\ \eta = h_1}}. \tag{48}$$

These results follow directly from the Poisson formula that if the function δy receives a positive (negative) increment to the right (left) of the point a, then the right side of (48) diminishes. Analogously, if the function δy_0 receives a positive (negative) increment to the right (left) of the point a, then the right side of (48) also diminishes. Therefore, we obtain a lower bound for $\delta x''(a)$ if we replace the function δy in (48) by $v(\xi, \eta)$, which exceeds the value of $\rho : \rho(\xi) = \delta \bar{y}(a) + \bar{q}(\xi - a)$ on the line $\eta = h_1$ when ξ belongs to the interval γ: $|\delta \bar{y}(a) + q(\xi - a)| < m(\xi)$; $\rho = m(\xi)$ when ξ is to the right of γ and $\rho = -m(\xi)$ when ξ is to the left of γ, and the value of ρ on the line $\eta = 0$ is such that $\int_a^\xi \rho d\xi = m_0(\xi)(\xi - a)$ for $\xi > a$ and $\int_a^\xi \rho d\xi = -m_0(\xi)(\xi - a)$ for $\xi^a < a$.

Let us represent the function v by the sum

$$v(\xi, \eta) = v_1(\xi, \eta) + v_2(\xi, \eta),$$

where v_1 and v_2 are harmonic functions defined by the conditions

$$v_1(\xi, h_1) = v(\xi, h), \; v_1(\xi, 0) = 0,$$
$$v_2(\xi, h_1) = 0, \; v_2(\xi, 0) = v(\xi, 0).$$

By virtue of the reasoning presented above, at the point a we have

$$\delta x''(a) > \frac{\partial}{\partial \eta}\left(\frac{\partial v_1}{\partial \xi}\right) + \frac{\partial^2 v_2}{\partial \xi \partial \eta}. \tag{49}$$

We obtain for the first component by means of the Poisson formula

$$\frac{\partial}{\partial \eta}\left(\frac{\partial v_1}{\partial \xi}\right) \geqslant \frac{q}{h_1} + \frac{\pi}{h_2^2} \int_{\frac{2\varepsilon_1}{q}}^{\infty} \frac{q - m'(\xi - a)}{\mathrm{sh}^2 \frac{\pi}{2} \frac{\xi}{h_1}} d\xi = \frac{q}{h_1} + \frac{\pi q}{h_1^2} \int_{\frac{2\varepsilon_1}{q}}^{\frac{1}{k\sqrt{h}} - a} \frac{d\xi}{\mathrm{sh}^2 \frac{\pi}{2} \frac{\xi}{h_1}} + \pi \frac{q - 2k\sqrt{h\varepsilon}}{h^2} \int_{\frac{1}{k\sqrt{h}} - a}^{\infty} \frac{d\xi}{\mathrm{sh}^2 \frac{\pi}{2} \frac{\xi}{h_1}}$$

or, taking into account that $q > \varepsilon$, and integrating, we obtain

$$\frac{\partial}{\partial \eta} \frac{\partial v_1}{\partial \xi} \geqslant \frac{q}{h} + \frac{kq}{h}\left[\mathrm{cth} \frac{\pi \varepsilon_1}{qh} - 1\right]. \tag{50}$$

Applying the same Poisson formula for the second number in (49), integrating by parts, and taking account of (41"), we obtain

$$\left|\frac{\partial^2 v_2}{\partial \xi \partial \eta}\right| < \frac{k}{h^3} \int_0^\infty \frac{m_0(\xi - a)|\xi - k\sqrt{h}|\, d\xi}{\mathrm{ch}^2 \frac{\varepsilon}{h_1}} < \frac{k\varepsilon_0}{h^{\frac{3}{2}}} + kh\varepsilon. \tag{51}$$

Comparing (47), (49), (50), and (51), we obtain

$$\frac{q}{h}\left[\mathrm{cth} \frac{\pi \varepsilon}{qh} - 1\right] - Qq - \frac{k\varepsilon_0}{h^{3/2}}.$$

Therefore, either

$$q < \frac{k\varepsilon_0}{h^{3/2}},$$

or

$$\operatorname{cth}\frac{\pi\varepsilon}{qh}-1<\theta h,$$

i.e.,

$$q<\frac{k\varepsilon}{h\log\frac{1}{h}}.$$

We obtain the desired estimate if we also use (43).

LEMMA 8. For $\varepsilon > \theta\varepsilon_0/h^{3/2}$ we have

$$|\delta y''|=|y_1''(x)-y''(x)|<\frac{k\varepsilon}{h}. \tag{52}$$

Proof. First, let us go from the variation $\delta y''$ to the variation $\delta\bar{y}''$. In conformity with (41), (46), and (47), we have

$$\begin{gathered}\delta\bar{y}'=y'(x_1)x_1'-y'(x)x'+\delta y'x_1', \\ \delta\bar{y}''=y''(x_1)x_1'^2-y''(x)x'^2+y'(x_1)x_1''+\delta y'x_1''-y'(x)x''+\delta y''x_1'^2= \\ =y'''x'^2\delta x+2y''x'\delta x'+y''x''\delta x+y'\delta x''+\delta y'x_1''+\delta y''x_1'^2\ldots\end{gathered} \tag{53}$$

Hence, taking into account (41) and Lemmas 2-6, we obtain

$$\delta\bar{y}''=(1+\theta h)\,\delta y''+\frac{k\varepsilon}{h\log\frac{1}{h}}.$$

The problem is thereby reduced to estimating $\delta\bar{y}''$.

Retaining the method presented to prove Lemma 6, let us consider the function $\delta\bar{y}''$ on the segment $[-\xi_1, \xi_1]$, $\xi_1 = k\sqrt{h}$ and let a_1 denote the point at which $|\delta\bar{y}''|$ reaches the absolute maximum; let the maximum for $\delta\bar{y}''$ hence be reached at a and let it not be less than $k\varepsilon/h$. In this case it is seen that there is a point a in the segment $[a_1, a_1+\sqrt{h/k}]$ such that for any ξ we will have

$$-u(\xi)=\delta\bar{y}(\xi)-\delta\bar{y}(a)-\delta\bar{y}'(a)(\xi-a)-\tfrac{1}{2}\,\delta\bar{y}''(a)(\xi-a)^2>0,$$

and, moreover,

$$q=\delta\bar{y}''(a)\geqslant\delta\bar{y}''\ (a_1).$$

Now, let us express $\delta x'(a)$ in terms of values of $\delta\bar{y}(\xi)$ and $\delta\bar{y}_0(\xi)$. To this end, let $v_1(\xi, \eta)$ and $v_2(\xi, \eta($, respectively, denote harmonic functions defined by the boundary conditions

$$\begin{gathered}v_1(\xi,\ h)=\delta\bar{y}(\xi),\ v_1(\xi,\ 0)=0, \\ v_2(\xi,\ h)=0,\ v_2(\xi,\ 0)=\delta\bar{y}_0(\xi).\end{gathered}$$

We have

$$\delta x'(a)=\left[\frac{\partial v_1}{\partial\eta}\right]_{\substack{\xi=a\\ \eta=h}}+\left[\frac{\partial v_2}{\partial\eta}\right]_{\substack{\xi=a\\ \eta=h}}. \tag{54}$$

Using the Poisson integral, we obtain

$$\frac{\partial v_1}{\partial\eta}=\frac{\delta\bar{y}}{h}\frac{\pi}{4h^2}\int\limits_{-\infty}^{\infty}\frac{\delta\bar{y}''(a)(\xi-a)^2}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi-a}{h}}\,d\xi-\frac{\pi}{h^2}\int\limits_{-\infty}^{\infty}\frac{u(\xi)\,d\xi}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi-a}{h}}=\frac{\delta\bar{y}}{h}+khq-\frac{\pi}{h^2}\int\limits_{-\infty}^{\infty}\frac{u(\xi)\,d\xi}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi-a}{h}}. \tag{55}$$

In conformity with (43), for $|\varepsilon_1| < \theta\sqrt{h}$ we obtain

$$\left|\frac{\partial v_2}{\partial \eta}\right| < \frac{\theta}{h^3} \int\limits_{-\infty}^{\infty} \frac{\varepsilon_0 |\xi| + kh^{3/2}\varepsilon\xi^2}{\mathrm{ch}^2 \frac{\pi}{2} \frac{\xi - a}{h}} d\xi < \theta\varepsilon. \tag{56}$$

Let us compare (46), (54), (55), and (56). According to (46) and Lemma 6

$$\delta x'(a) = \frac{\delta(\bar{y})}{h} + \theta\varepsilon. \tag{57}$$

If from the opposite

$$q > \frac{K\varepsilon}{h},$$

where K is selected sufficiently large as a function of k and θ, and if

$$\varepsilon_0 < \theta h \varepsilon_1,$$

then by virtue of (54), (55), and (56) we obtain

$$\delta x'(a) = \frac{\delta\bar{y}}{h} + Kk\varepsilon \frac{\theta}{h^2} \int\limits_{-\infty}^{\infty} \frac{u(\xi)\, d(\xi)}{\mathrm{sh}^2 \frac{\pi}{2} \frac{\xi - a}{h}}. \tag{58}$$

Therefore, by equating (57) and (58), we obtain

$$\frac{1}{h^2} \int\limits_{-\infty}^{a} \frac{u(\xi)\, d(\xi)}{\mathrm{sh}^2 \frac{\pi}{2} \frac{\xi - a}{h}} > \theta h q. \tag{59}$$

Hence, in order to arrive at a contradiction, let us evaluate $\delta x'''(a)$. Differentiating (46) and noting that $\delta\bar{y}'''(a)$, we obtain

$$\delta x'''(a) = \frac{q}{h}(1 + \theta h) + \frac{\theta\varepsilon}{h \log \frac{1}{4}}. \tag{60}$$

Let us express this same quantity in terms of values of $\delta\bar{y}''(\xi)$ and $\delta\bar{y}_0''(\xi)$ by means of the Poisson formula. Using the functions $v_1(\xi, \eta)$ and $v_2(\xi, \eta)$ introduced above, we obtain

$$\delta x'''(a) = \left[\frac{\partial^3 v_1}{\partial\eta\partial\xi^2}\right]_{\substack{\xi=a\\ \eta=h}} + \left[\frac{\partial^3 v_2}{\partial\eta\partial\xi^2}\right]_{\substack{\xi=a\\ \eta=h}}. \tag{61}$$

We have for the first member

$$\frac{\partial}{\partial\eta}\frac{\partial^2 v_1}{\partial\xi^2} = \frac{h}{q} + \frac{\pi}{2h^2} \int\limits_{-\infty}^{\infty} \frac{u''(\xi)\, d\xi}{\mathrm{sh}^2 \frac{\pi}{2} \frac{\xi - a}{h}}, \tag{62}$$

and by using (43) for the second we obtain

$$\left|\frac{\partial^2}{\partial\xi^2}\frac{\partial v_2}{\partial\eta}\right| < \frac{\theta\varepsilon}{h^5} \int\limits_{-\infty}^{\infty} \frac{|\xi - a| + kh^{3/2}}{\mathrm{ch}^2 \frac{\xi - a}{h}} d\xi < \frac{\theta\varepsilon}{h^{3/2}}. \tag{63}$$

Assuming q to be large compared with ε/h and using the condition

$$\varepsilon_0 < Qh^{3/2}\varepsilon,$$

from a comparison of (60), (61), (62), and (63) we obtain

$$\frac{1}{h^2}\int_{-\infty}^{\infty}\frac{u''(\xi)\,d\xi}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi-a}{h}} < \frac{1}{K}\frac{q}{h}, \tag{64}$$

where K can be taken arbitrarily large. Integrating (64) twice by parts, we obtain

$$\frac{0}{h^2}\int_{-\infty}^{\infty}\frac{u(\xi)\,d\xi}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi-a}{h}} < \frac{hq}{K},$$

which contradicts (59) for large K.

The following refinement of Lemma 6 follows directly from what has been proved.

LEMMA 9. Under the conditions of Lemma 7 we have

$$|\delta y'| = |y_1'(x) - y'(x)| < \frac{\bar{k}\varepsilon}{h^{1/2}}. \tag{65}$$

LEMMA 10. For $\varepsilon_0 < kh^2\varepsilon$

$$|\delta y'''| = |y_1'''(x) - y'''(x)| < \frac{\bar{k}\varepsilon}{h^2\log\frac{1}{h}}. \tag{66}$$

Proof. In order to be able to perform a calculation in the variables ξ, η, let us find an estimate for $Q''' = \partial^3 Q/\partial\xi^3$ for $\eta = h$. Starting from (24) and (25), and having estimates for Q, Q', and Q'', it is easy to see that $|P''| < k$, $|P'''| < k/h\log(1/h)$ and $Q''' = \bar{y}^{IV}(1 + \theta h)$; hence, taking (24) into account, we obtain

$$P^{IV} = \frac{\theta Q'''}{h}. \tag{67}$$

Taking into account that the function $\partial^2\theta/\partial\xi^2$ is conjugate to the function $\partial^2 P/\partial\xi^2$ for $\eta = h$ we have

$$Q''' = \frac{\partial P''}{\partial\eta}.$$

Let us construct the harmonic functions P_0'' and P_1'':

$$P_0'' = \begin{cases} P'' & \text{for} \quad \eta = h, \\ 0 & \text{for} \quad \eta = 0; \end{cases}$$

$$P_1'' = \begin{cases} 0 & \text{for} \quad \eta = h, \\ P'' & \text{for} \quad \eta = 0; \end{cases}$$

we evidently have

$$Q''' = \frac{\partial P_0''}{\partial\eta} + \frac{\partial P_1''}{\partial\eta}. \tag{68}$$

Let ξ_0 denote the point at which $|Q'''|$ reaches an absolute extremum and let us estimate each of the members on the right separately. We have

$$\left|\frac{\partial P_0''}{\partial \eta}\right| < \frac{(P''\xi_0)}{h} + \frac{\pi}{2}\frac{1}{h^2}\int_{-\infty}^{\infty}\frac{P''(\xi) - P''(\xi_0) - (\xi - \xi_0)P'''(\xi_0)}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi-\xi_0}{h}}d\xi + \frac{\pi}{2}\frac{1}{h^2}\int_{|\xi-\xi_0|>a}\frac{(\xi-\xi_0)P'''(\xi_0)\,d\xi}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi-\xi_0}{h}}. \tag{69}$$

The last integral is evidently zero for any $a > 0$. Noting that

$$|P''(\xi) - P''(\xi_0)| < k, \tag{70}$$

let us set

$$a = \sqrt{\frac{kh}{Q'''(\xi_0)}};$$

then (69) can be rewritten as

$$\left|\frac{\partial P_0''}{\partial \eta}\right| < \frac{k}{h} + \frac{\pi}{2}\frac{1}{h^2}\int_{\xi_0-a}^{\xi_0+a}\frac{|P^{\mathrm{IV}}|(\xi-\xi_0)^2}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi-\xi_0}{h}}d\xi + \frac{\pi}{2}\frac{k}{h^2}\int_{|\xi-\xi_0|>a}\frac{d\xi}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi-\xi_0}{h}} < \frac{k}{h} + k\sqrt{\frac{Q'''(\xi_0)}{h}}. \tag{71}$$

Let us turn to an estimation of the second member in (68). We have

$$\frac{\partial P_1''}{\partial \eta} = \frac{k}{h^2}\int_{-\infty}^{\infty}\frac{P''(\xi, 0) - d\xi}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi-\xi_0}{h} + k} = \frac{k}{h^3}\int_{-\infty}^{\infty}\frac{P(\xi_1, 0) - P'(\xi_0, 0)}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi-\xi_0}{h}}. \tag{72}$$

Let us estimate $|P'(\xi, 0) - P'(\xi_0, 0)|$ by considering P as a function conjugate to Q,

$$P'(\xi, 0) = \frac{1}{h}\{Q(\xi, h) - Q(\xi, 0)\} +$$
$$+ \frac{\pi}{2}\frac{1}{h^2}\int_{-\infty}^{\infty}\frac{Q(t, 0) - Q(\xi, 0) - (t-\xi)Q'(\xi, 0)}{\mathrm{sh}^2\frac{\pi}{2}\frac{t-\xi}{h}}dt +$$
$$+ \frac{\pi}{2}\frac{1}{h^2}\int_{-\infty}^{\infty}\frac{Q(t, h) - Q(\xi, h) - (t-\xi)Q'(\xi, h)}{\mathrm{sh}^2\frac{\pi}{2}\frac{t-\xi}{h} + k}dt.$$

Therefore, taking the explicit estimates for Q and their derivatives into account, we obtain

$$|P'(\xi, 0) - P'(\xi_0, 0| < k\,|\xi - \xi_0| + kh;$$

but then (72) will yield

$$\left|\frac{\partial P_1''}{\partial \eta}\right| < \frac{k}{h}. \tag{73}$$

Now comparing (63), (71) and (73), we obtain

$$Q'''(\xi_0) < \frac{k}{h} + k\sqrt{\frac{Q'''(\xi_0)}{h}},$$

which finally yields

$$|Q'''| < \frac{k}{h}. \tag{74}$$

By virtue of the estimates obtained earlier, $\delta Q''$ can be represented as

$$\delta Q'' = \delta y''' + \frac{k\varepsilon}{h\log\frac{1}{h}} + Q'''\delta x,$$

from which, by taking account of (42) and (74), we obtain

$$|\delta y'''| = |\delta Q''| + \frac{k\varepsilon}{h^2}(Qh + \xi). \tag{75}$$

This means that for pur purposes it is sufficient to prove that

$$|\delta Q''| < \frac{k\varepsilon}{h^2 \log \frac{1}{h}}. \tag{76}$$

From the opposite, let us assume that there is a point on the segment $|\xi| < k\sqrt{y}$ at which

$$|\delta Q''| > \frac{K\varepsilon}{h^2 \log \frac{1}{h}},$$

where K is large together with 1/h; but then according to Lemma 7, $|\delta Q'| < \theta\varepsilon/h$ and there is a point ξ_0 in the interval $|\varepsilon| < 2k\sqrt{h}$ such that

$$|\delta Q''(\xi_0)| > -\frac{K\varepsilon}{h^2 \log \frac{1}{h}}, \quad \delta Q'''(\xi_0) = 0;$$

hence, either

$$\delta Q'(\xi) \begin{cases} \leqslant \delta Q'(\xi_0) + \delta Q''(\xi_0)(\xi - \xi_0), & \xi > \xi_0, \\ \geqslant \delta Q'(\xi_0) + \delta Q''(\xi_0)(\xi - \xi_0), & \xi < \xi_0, \end{cases}$$

or

$$\delta Q'(\xi) \begin{cases} \geqslant \delta Q'(\xi_0) + \delta Q''(\xi_0)(\xi - \xi_0), & \xi > \xi_0, \\ \leqslant \delta Q'(\xi_0) + \delta Q''(\xi_0)(\xi - \xi_0), & \xi < \xi_0. \end{cases}$$

For definiteness, let us investigate the first case. By virtue of (24)

$$\delta P''' = \frac{\delta \bar{y}'''}{h}(1 + \theta h) = \frac{\delta Q''}{h}(1 + \theta h) + \frac{k\varepsilon}{h \log \frac{1}{h}} \tag{77}$$

or setting $q = \delta Q''(\xi_0)$ and taking (75) into account,

$$\delta P''' = \frac{q}{h}(1 + \theta h).$$

Let us find $\delta P'''$ by considering $\delta P''$ as a function conjugate to $\delta ''Q$. We have

$$\delta P'''(\xi_0) = \frac{\partial}{\partial \eta}\delta Q''. \tag{78}$$

In the strip $0 < \eta < h$ let us construct harmonic functions $u_1(\xi, \eta)$ and $u_0(\xi, \eta)$ with the boundary conditions

$$u_1(\xi, h) = \int_0^{\xi} \delta Q(\xi, h)\, d\xi, \quad u_1(\xi, 0) = 0,$$

$$u_2(\xi, h) = 0, \quad u_2(\xi, 0) = \int_0^{\xi} \delta Q(\xi, 0)\, d\xi.$$

In conformity with (78), we have

$$\delta P'''(\xi_0) = \frac{\partial u_1'''}{\partial \eta} + \frac{\partial}{\partial \eta}\frac{\partial^3 u^2}{\partial \xi^3}. \tag{79}$$

Let us estimate each of the members in the right side of (79) separately.

For the first member we have

$$\frac{\partial u_1'''}{\partial \eta} = \frac{q}{h} + \frac{\pi}{2}\frac{1}{h^2}\int_{-\infty}^{\infty}\frac{q-\delta Q''}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi-\xi_0}{h}}\,d\xi. \tag{80}$$

In conformity with the reasoning which has been used repeatedly above, and taking into account that $|\delta Q'| < k\varepsilon/h$, we obtain the least value for the integral in the right side of (80)if we set $\delta Q'' = q$ for $|\xi - \xi_0| < k\varepsilon_1/qh$ and $\delta Q'' = 0$ for the remaining values of ξ. Therefore,

$$\frac{\partial u_1'''}{\partial \eta} \geqslant \frac{q}{h} + \frac{\pi q}{h^2}\int_{\frac{\varepsilon}{qh}}^{\infty}\frac{d\xi}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi}{h}} = \frac{q}{h} + \frac{2q}{h}\left[\mathrm{cth}\frac{\varepsilon}{qh^3} - 1\right]. \tag{81}$$

Let us examine the other members in (79). We have

$$\left|\frac{\partial}{\partial \eta}\frac{\partial^3 u_2}{\partial \xi^3}\right| < \frac{k}{h^5}\int_{-\infty}^{\infty}\frac{|u_2(\xi,0)|\,d\xi}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi-\xi_0}{h}+k} = \frac{\theta}{h^6}\int_{-\infty}^{\infty}\frac{\int_{\xi_0}^{\xi}|u_2|\,d\xi}{\mathrm{sh}^2\frac{\pi}{2}\frac{\xi-\xi_0}{h}+k}. \tag{82}$$

Let us obtain the estimate of $\int_{\xi_0}^{\xi}|u_2|\,d\xi$. In conformity with giving u_2, by assuming $\xi_0 \geq 0$, we have

$$v = \int_{\xi_0}^{\xi}|u_2|\,d\xi \leqslant \int_0^{\xi}|u_2|\,d\xi = \int_0^{\xi}d\xi\left|\int_0^{\xi}\delta Q(\xi,0)\,d\xi\right|.$$

For the subsequent calculation we consider that by retaining just the principal terms we have

$$\bar{y}' = \mathrm{tg}\,Q e^{P} = Q + QP,$$
$$\delta\bar{y}' = \delta Q(1+P) + Q\delta P,$$
$$\delta\bar{y}'(\xi)\,\delta\bar{y}(0) = u_2 + \int_0^{\xi}P\delta Q\,d\xi + \int_0^{\xi}Q\delta P\,d\xi$$

or

$$v \leqslant \int_0^{\xi}|\delta\bar{y}(\xi) - \delta\bar{y}(0)|\,d\xi + \int_0^{\xi}d\xi\left|\int_0^{\xi}P\delta Q\,d\xi\right| + \int_0^{\xi}d\xi\left|\int_0^{\xi}Q\delta P\,d\xi\right|.$$

Considering that $|Q(\xi, 0)| < kh^{5/2}$ and $|Q'(\xi, 0)| < kh^2$ for the last two integrals, we have

$$\int_0^{\xi}d\xi\left|\int_0^{\xi}P\delta Q\,d\xi\right| \leqslant \int_0^{\xi}|Pv|\,d\xi + \int_0^{\xi}d\xi\int_0^{\xi}|P'v|\,d\xi < kh^{1/2}v,$$

$$\int_0^{\xi}d\xi\left|\int_0^{\xi}Q\delta P\,d\xi\right| \leqslant \int_0^{\xi}|Q|\,d\xi\left|\int_0^{\xi}\delta P\,d\xi\right| + \int_0^{\xi}d\xi\left|\int_0^{\xi}Q''\,d\xi\int_0^{\xi}\delta P\,d\xi\right| \leqslant$$

$$\leqslant kh^{5/2}\int_0^{\xi}|\delta x|\,d\xi + kh^2\int_0^{\xi}d\xi\int_0^{\xi}|\delta x|\,d\xi \leqslant kh^{5/2}\varepsilon\xi^2 + kh^{5/2}\varepsilon_0|\xi| + kh\varepsilon|\xi|^3 + kh^2\varepsilon_0\xi^2.$$

This means that taking account of (42') and the condition $|\xi_0| < k\sqrt{h}$,

$$v < k\varepsilon_0|\xi| + kh^{5/2} + kh^2|\xi-\xi_0| + kh^{3/2}|\xi-\xi_0|^2 + kh|\xi-\xi_0|^3.$$

Substituting the expression found for v into (82), we finally obtain

$$\left|\frac{\partial}{\partial\eta}\frac{\partial^3 u_2}{\partial\xi^3}\right| < k\varepsilon_0 h^{-9/2} + k\varepsilon h^{-5/2} < k\varepsilon h^{-5/2}. \tag{83}$$

Comparing (79), (81), and (83), we obtain

$$\delta P'''(\xi_0) > \frac{q}{h} + \frac{2q}{h}\left[\operatorname{cth}\frac{\varepsilon}{qh^2} - 1\right] - k\varepsilon h^{-5/2}.$$

Comparing this inequality to (77) and taking (75) into account, we find

$$\operatorname{cth}\frac{\varepsilon}{qh^2} - 1 < \theta h + \frac{h^{1/2}}{K}\log\frac{1}{h} < kh^{1/2}$$

or

$$q < \frac{k\varepsilon}{h^2 \log\frac{1}{h}},$$

which contradicts (75).

9. ESTIMATE OF THE DERIVATIVE OF THE REMAINDER TERM

Starting from the estimates obtained above for variations of the derivatives of the wave line, let us estimate the variation of the remainder term in the approximate formula (2) upon going from this wave to an infinitely close by wave. Using the notation (1), (17), let us assume

$$R = I(\Gamma_0, \Gamma) - I(\Gamma_0, \Gamma) = |f'(z_1, \Gamma_0, \Gamma)| - |f'(z_1, z_1) + ky'^2 + z. \tag{84}$$

Let the function F (z) (F (± ∞) = z_1), F (± ∞) = z_1) which realizes the conformal mapping of the domain D(Γ_0, Γ) onto the domain D (C_0, C) upon correspondence between the infinitely remote points of D (Γ_0, Γ) to the angular points of the crescent D (C_0, C). A line parallel to the x axis and passing through the point z_0 is taken as C_0. We evidently have

$$f'(z_1, \Gamma_0, \Gamma) = f'(z_1, z_1 F'(z_1))$$

or retaining only the principal terms,

$$|f'(z_1, \Gamma_0, \Gamma)| = |f'(z_1, z_1)|^2 + 2|f'(z_1, z_1)|^2 \log|F'(z_1)|;$$

hence,

$$R = 2|f'(z_1, z_1)|^2 \log|F'(z_0)| + ky'^2 + r. \tag{85}$$

LEMMA 11. Under the conditions of Lemma 10, we have

$$|\delta R| < \frac{\bar{k}\varepsilon}{\log\frac{1}{h}}, \tag{86}$$

where $\bar{k}$ is a constant dependent only on the constant k in the conditions of Lemmas 7-10.

Proof. In conformity with (85), we have

$$\delta R = k\frac{\delta|F'|}{|F'|} + 4|f'|\log|F'|\,\delta|f'| + ky'\delta y' + \delta r, \tag{87}$$

where we consider the variation for a fixed value of $x = x_1$, $|x_1| < k\sqrt{h}$. In conformity with the estimates obtained above, the last three members in (87) will be on the order of $h\varepsilon$, which means that it is sufficient to estimate the first member. Let $\bar{C}$ denote the contour which passes through the point $\bar{z}_1 = x_1 + i[y(x_1) + \delta y(x_1)]$ and is tangent to $\bar{\Gamma}$ at this point, and let C_0 be a line parallel to the x axis and separated from it by the spacing $\delta y_0(x_1)$

$$|\delta y_0(x_1)| \leqslant \varepsilon' + \varepsilon_0.$$

Let us map the domain $D(\bar{C}_0, \bar{C})$ conformally into the domain $D(C_0, C)$ under the condition of correspondence between the angular points, as well as the points z_1, $\bar{z}_1$. Hence, let the line $\bar{\Gamma}$ go over into the line γ and the line $\bar{\Gamma}_0$ into the line γ_0. Let us map conformally $z' = y(z)$, $\varphi(\pm\infty) = \pm\infty$, $\varphi(z_1) = z_1$ the domain $D(\Gamma_0, \Gamma)$ into the domain $D(\gamma_0, \gamma)$. By virtue of the elementary rule of differentiating complex functions, we obtain

$$\frac{\delta|F'|}{|F'|} = \log|\varphi'(z_1)|. \tag{88}$$

Let $\rho = \rho(x)$ and $\rho_0 = \rho_0(x)$, respectively, denote the differences between ordinates of points of the lines γ, Γ and γ_0, Γ_0. Let us estimate each of these functions separately. To this end, let $u(x)$, $u_0(x)$ denote the difference between ordinates of the points C, Γ and C_0, Γ_0, $\bar{u}(x)$ and $\bar{u}_0(x)$ the difference between ordinates of the points $\bar{C}$, $\bar{\Gamma}$ and $\bar{C}_0$, $\bar{\Gamma}_0$. By virtue of Lemma 9 we have

$$|\bar{u}(x) - u(x)| < \frac{k\varepsilon}{h^2 \log\frac{1}{h}} |x - x_1|^3$$

and by assumption

$$|\bar{u}_0(x) - u_0(x)| < \varepsilon_0 < kh^2\varepsilon.$$

Hence, taking Lemmas 7 and 8 into account, we obtain

$$\rho = \bar{u}(x) - (1+\mu)\, u\,(x + \nu x),$$

where $|\mu|$ and $|\nu|$ do not exceed $k\varepsilon/h$, i.e.,

$$|\rho| < \frac{k\varepsilon}{h^2 \log\frac{1}{h}} |x - x_1|^3 + \frac{\varepsilon}{h} u + \frac{\varepsilon}{h} u' |x - x_1|,$$

or noting that $u = \int\int\int y''' dx^3$ and using Lemma 5,

$$|\rho| < \frac{k\varepsilon}{h^2 \log\frac{1}{h}} |x - x_1|^3.$$

Perfectly analogously, for the function $\rho_0(x)$ we obtain

$$|\rho_0| = |\bar{u}_0(x) - (1+\mu)\, u_0(x + \delta x)| < \varepsilon_0 + \mu|u_0| + k u_0' |\delta x|,$$

where the function δx satisfies the inequality

$$\int_{x_1}^{x} |\delta(x)|\, dx < \frac{k\varepsilon}{h}(x - x_1)^2 + k\varepsilon_0 |x - x_1|$$

by virtue of (41').

Now, let us use the Poisson formula. In conformity with the definition of the function $y(z)$, we shall have

$$|\log|\varphi'(z_1)|| < \frac{k}{h^2} \int_{-\infty}^{\infty} \frac{|\rho|\, dx}{\mathrm{sh}^2 \frac{\pi}{2}\frac{x - x_1}{h}} + \frac{k}{h^2} \int_{-\infty}^{\infty} \frac{|\rho_0|\, dx}{\mathrm{sh}^2 \frac{\pi}{2}\frac{x - x_1}{h} + k} < \frac{k\varepsilon}{\log\frac{1}{h}}.$$

10. CONTINUITY OF THE OPERATOR $H\{y_0\}$ AND THE FUNCTIONAL $\Delta\{y_0\}$

Let us turn to the main problem formulated in Sec. 6. Let us assume that for some class $\{y_0(x)\}$ of lines with period 2ω, the solution of this problem $y = y(x) = H\{y_0\}$, $y(0) = h + \alpha$ exists. In this section we shall show that this solution depends continuously on $y_0(x)$ and we also establish the degree of this continuity.

LEMMA 12. Let $|\delta y_0(x)| < \varepsilon_0$, $\Delta\{y_0\} = 0$ and let $y_0(x)$ satisfy the conditions in Sec. 8 and such that

$$|Y_0(x) - y_0(x)| < \nu h^3, \tag{89}$$

where ν tends to zero together with h and (11) has an integral of the function $Y(x, \alpha)$ denoted in Sec. 4, for $\eta = Y_0(x) - h$, where $2h^2 > \alpha > 2h - \mu$, $\lim \mu = 0$; then for a sufficiently small ε_0 we obtain

$$|\delta H\{y_0\}| = |\{y_0 + \delta y_0\} - H\{y_0\}| < \frac{k\varepsilon_0}{h^2}, \tag{90}$$

$$|\delta\Delta\{y_0\}| < \frac{k\varepsilon_0}{h}. \tag{90'}$$

Proof. Let us show first that for $|x| < k\sqrt{h}$ we have

$$|Y - y| < k\nu h^2. \tag{91}$$

Indeed, along the line $\Gamma: y = (x)$

$$I(\Gamma_0, \Gamma) = I(\Gamma_0, \Gamma) + R = 0;$$

but in conformity with (5) and Lemma 6,

$$|R| < \frac{kh^2}{\log\frac{1}{h}},$$

from which we obtain the desired inequality for $|x| < k\sqrt{h}$ by applying Lemma 2.

Hence, taking account of the properties of the function $Y(x, \alpha)$ noted in Sec. 4 for α close to $2h^2$, for the function $y(x)$ for $x\sqrt{h} < x < \omega$ we have

$$y(x) < h - \theta h^2,\ 0 < \theta < 1. \tag{92}$$

Now, let us prove that if

$$|\Delta| > \frac{k\varepsilon_0}{h},$$

then

$$|\delta y| < \frac{k|\Delta|}{h}. \tag{93}$$

For definiteness, let us take $\Delta < 0$ and let us assume the opposite, i.e., that

$$|\delta y| > \frac{K|\Delta|}{h},$$

where K can be taken arbitrarily large. Let k_0 denote the point at which $|\delta y|$ reaches its greatest value and let us examine two cases separately: a) $x_0 < k\sqrt{h}$, b) $x_0 \geq k\sqrt{h}$.

Let us start with the first case.

Let $|\delta y(x_0)| = K_0|\Delta|/h$, $K_0 \geq K$. At the point x_0 we have

$$\delta J = \delta I + \delta R = 0, \tag{94}$$

where in conformity with Lemma 2,

$$|\delta I| > \frac{K_0 - K}{h}|\Delta|, \tag{95}$$

and in conformity with Lemma 11,

$$|\delta R| < \frac{k|\delta y(x_0)|}{\log\frac{1}{h}} = \frac{kK_0|\Delta|}{h\log\frac{1}{h}}. \tag{96}$$

Comparing (94), (95), and (96), we arrive at a contradiction.

Now, let us examine case b) when $x_0 \geq k\sqrt{h}$. For definiteness, let us assume that $\delta y(x_0) = |\delta y(x_0)| = K_0|\Delta|/h$. As in case a), we have the relationship (94). Let us estimate δI at the point x_0. By virtue of (92) and the condition

$$\delta y''(x_0) \leqslant 0$$

we find

$$\delta I < \frac{6K_0\theta - k}{h}\Delta,$$

which contradicts the relations (94) and (96). The case when $\delta y(x_0) < 0$ at the point x_0 is considered perfectly analogously. The inequality (93) is thereby proved completely. If $|\Delta| < k\varepsilon_0/h$, then we show by analogous reasoning that $|\delta y| < k\varepsilon_0/h^2$. This means that there remains to estimate $|\Delta|$. Let us again limit ourselves to the case $\Delta = -\Delta_1 < 0$. Let us assume the opposite, i.e., that

$$\Delta_1 > \frac{k\varepsilon_0}{h},$$

where K is arbitrarily large. By the condition along the curves $\Gamma_0 : y = y_0(x)$ and $\Gamma : y = y(x)$ we have

$$I(\Gamma_0, \Gamma) + R = 0,$$

and along the curves $\Gamma_0 : y = y_0(x) + \delta y_0(x) - \Delta_1$ and $\Gamma : y = y(x) + \delta y(x)$

$$I(\bar{\Gamma}_0, \Gamma) + R + \delta R = 0,$$

where by virtue of (5), (93), and Lemmas 6 and 11 we have

$$|R| < \frac{kh^2}{\log\frac{1}{h}};$$

$$|\delta R| < \frac{k\Delta_1}{h\log\frac{1}{h}}.$$

Hence, considering the variation δI and taking account of (91), we obtain

$$\delta y'' = -\frac{9}{h^3}\left(Y - h - \frac{1}{3}v + \frac{\theta h^2}{\log\frac{1}{h}}\right)\delta y + \frac{3\Delta}{h^2} + \frac{k\Delta}{h^2\log\frac{1}{h}}.$$

Hence, by virtue of Lemma 1 for sufficiently small h and for $0 < x < 2k_0\sqrt{h}$ we will have

$$\delta y' > 0,$$

where

$$\delta y(x_1) = \frac{k\Delta_1}{h}.$$

for $x_1 = k_0\sqrt{h}$. Here k_0 is fixed so that (92) would hold for $x > x_1$.

Let us consider the parabola

$$\varphi = \frac{k_0\Delta_1}{h}\frac{(x-\omega)^2}{(\omega-x_1)^2}$$

and let x_0 denote the point at which the difference $\delta y - \varphi$ reaches its greatest value. We obtain in the construction.

$$x_0 > x_1, \quad y(x_0) < h - \theta h^2, \quad \delta y(x_0) > \frac{k_0\Delta_1}{h},$$
$$\delta y - \varphi - [\delta y(x_0) - \varphi(x_0)] \leqslant 0.$$

Noting this, let us again examine the variation δI at the point x_0; for sufficiently small h

$$\delta I > \frac{k\Delta_1}{h};$$

however, in the case considered

$$|\delta R| < \frac{k\Delta_1}{h\log\frac{1}{h}},$$

and we have again arrived at a contradiction to (94). This means $\Delta_1 < k\varepsilon_0/h$, which together with (93) proves the lemma completely. Let us prove still another assertion which yields an estimate $\Delta\{y_0\}$ for a special class of lines y_0.

LEMMA 13. For $\eta = y_0(x) - v$ let the integral of (11) be the function $Y(x, \alpha)$, $\alpha = 2h^2 - \nu h^2$ while for $y = \bar{y}_0(x) - v$ the very same integral is the function $Y(x)$, $Y(0) = Y(0, x)$ and the period of Y agrees with the period of $Y(x, \alpha)$. Moreover, $\nu = kh^2$, and $|\bar{y}_0(x) - y_0(x)| < \nu h^3$, $|\Delta\{\bar{y}_0\}| < \mu h^3$. Under these conditions, if the numbers ν and μ are sufficiently small, we obtain

$$|\Delta\{\bar{y}_0\}| < \frac{kh^3}{\log\frac{1}{h}};$$
$$|H\{\bar{y}_0\} - Y(x)| < \frac{kh^2}{\log\frac{1}{h}}.$$

Proof. It can be seen that it is sufficient to examine the case when $\bar{y}_0(x) \equiv y_0(x)$. Moreover, let us consider $\Delta = -\Delta_1 < 0$, since the proof is perfectly analogous for $\Delta > 0$. Let us assume the opposite, i.e., that

$$\Delta_1 = \frac{K_1 h^3}{\log\frac{1}{h}}, \tag{97}$$

where K_1 is arbitrarily large. Let us construct the integral curve $\bar{\gamma} y = Y_1(x, \alpha)$ of (11) by setting $\eta = y_0(x) - v - \Delta_1$ in this equation. We obtain the expression

$$I(\bar{\Gamma}_0, \bar{\gamma})=0,$$

where $\bar{\Gamma}_0$ is the curve $y = y_0(x) - \Delta_1$. Moreover, by virtue of Lemma 6, along the line $\Gamma: y = H\{y_0\} = y\{x\}$

$$|I(\bar{\Gamma}_0, \Gamma)| < \frac{kh^2}{\log \frac{1}{h}}. \tag{98}$$

Therefore, by virtue of Lemma 2 for $|x| < x_1 = x\sqrt{h}$, we will have

$$|Y_1(x, \alpha) - y(x)| < \frac{k_1 h^2}{\log \frac{1}{h}}, \tag{99}$$

where x_1 can be chosen so that $Y(\frac{1}{2}x_0\alpha) < h - \theta h^2$. Moreover, by virtue of Lemma 1 and (97) for $x = \frac{1}{2}x_1$

$$\delta Y = Y_1\left(\frac{1}{2}x_1, \alpha\right) - Y\left(\frac{1}{2}x_1, \alpha\right) = \frac{k_2 K_1 h^2}{\log \frac{1}{h}}, \tag{100}$$

and for $x = x_1$

$$\delta Y = \frac{(k_2 + k_3) K_1 h^2}{\log \frac{1}{h}}.$$

Let us construct the parabola

$$\varphi = \varphi(x) = \frac{(k_1 + k_2) K_1 h^2}{\log \frac{1}{h}} \frac{(x - \omega)^2}{\left(\frac{1}{2}x_1 - \omega\right)^2}. \tag{101}$$

By virtue of (99) and (101), we obtain $y(x) < Y + \rho$ for $|x| \le \frac{1}{2}$, while

$$y(x_2) > Y(x_2, \alpha) + \varphi(x_2)$$

for $x = x_2$. This means that the difference $y(x) - Y(x, \alpha) - \varphi(x)$ will reach an absolute positive maximum at some point $x_0 > \frac{1}{2} x$, where at this point for sufficiently small μ we will have

$$y(x_0) < h - \theta h^2. \tag{102}$$

Let us introduce the curves $\Gamma_0: y = y_0(x)$ and $\gamma: y = Y(x, \alpha)$ into the considerations. By virtue of (4) and (16) at a point we obtain

$$J(\Gamma_0, \gamma) = I(\Gamma_0, \gamma) + kh^{\frac{5}{2}} = kh^{\frac{5}{2}}.$$

This means that at this same point

$$|J(\bar{\Gamma}_0, \gamma)| > \frac{\Delta_1}{h}(1 + \theta h) - kh^{5/2} > \frac{(K_1 - k) h^2}{\log \frac{1}{h}}.$$

But by virtue of (102) and the maximum condition $y - Y - \varphi$, at $x = x_0$ we obtain

$$|J(\bar{\Gamma}_0, \bar{\Gamma})| > J(\bar{\Gamma}_0, \gamma) - (1 + \theta h) h\varphi''(x_0) > \frac{(K_1 - k) h^2}{\log \frac{1}{h}},$$

which is impossible, since $J(\bar{\Gamma}_0, \bar{\Gamma}) = 0$ by assumption. The inequality (97) is thereby proved completely. However, the other desired inequality (98) results from (97) and Lemma 2 for $|x| < x\sqrt{h}$. For any x this inequality can be obtained if we use the relationships

$$y(x) < h - \theta h^2, \quad k\sqrt{h} < x < \omega,$$

by applying reasoning completely analogous to that presented in proving Lemma 11 and the first part of the lemma under consideration.

11. AUXILIARY ASSERTIONS

Let us note in addition several simple auxiliary assertions relative to the correspondence of boundaries for conformal mappings of domains which have a common part of a boundary.

Let the domain D be bounded by the lines $\Gamma_0 : y = y_0(x)$ and $\Gamma : y = y(x)$, where $y_0(x)$ satisfies condition (20) for $v = kh^2$, and $y(x)$ is such that

$$|y(x) - h| < kh^2, \quad |y'(x)| < kh^{3/2}, \quad |y''(x)| < kh.$$

Moreover, let there be given a line $\Gamma_1 : y = y_1(x)$ which separates the domain D into a domain $D_0 : y_0(x) < y < y_1(x)$ and $D_1 : y_1(x) < y < y(x)$ and hence such that

$$0 < \theta_0 h^3 < y_1(x) - y_0(x) < \theta_1 h^3.$$

Let us map the domains D_0 and D_1 conformally onto the strips $0 < \eta < \alpha h^3$, $\alpha h^3 < \eta < h_1 = h(1+\theta h)$ respectively, under the condition of correspondence between the infinitely remote points. Let $\zeta = f_0(z)$ and $\zeta = f_1(z)$ be functions which realize these mappings $f_0(\pm\infty) = \pm\infty$, $f_1(\pm\infty) = \pm\infty$.

As before, we let P_0 and P_1 denote $\log|f_0'(z)|$, $\log|f_1'(z)|$, respectively, and Q_0 and Q_1 are functions conjugate to P_0, P_1, $Q_0 = \arg f_0'(z)$, $Q_1 = \arg f_1'(z)$.

LEMMA 14. Let $y_1(x)$ have a continuous derivative, and let x_0 be a point at which $y_1'(x)$ reaches an absolute maximum (minimum); the function $y_1(x)$ is linear in the neighborhood of the point x_0. Under these conditions, if θ_0, $\theta_1 = k\alpha$, $|P_{0,1}| < k$ and

$$y_1'(x_0) > \max y_0'(x) + K\alpha h^{7/2} \left(y_1'(x_0) < \min y_0'(x) - K\alpha h^{7/2}\right),$$

then for $K > K_0(k)$ at point $[x_0, y_1(x_0)]$ line Γ_1 will have the form

$$\frac{d}{ds}(P_1 - P_0) < -kh^{\frac{1}{2}} \left(\frac{d}{ds}(P_1 - P_0) > kh^{\frac{1}{2}}\right),$$

where ds is an element of arc of the line Γ_1.

Proof. Indeed, in conformity with the definition of P, we have

$$\frac{dP_1}{ds} = \frac{\partial P_1}{\partial \xi} e^{P_1}, \quad \frac{dP_1}{\partial s} = \frac{\partial P_0}{\partial \xi} e^{P_0}.$$

At the same time

$$\frac{\partial P_1}{\partial \xi} = \frac{\partial Q_1}{\partial \eta} < \frac{kh^{3/2} + kh^{2+v}}{h_1} < kh^{1/2}.$$

$$\frac{\partial P_0}{\partial \xi} = \frac{\partial Q_0}{\partial \eta} < \frac{K\alpha h^{7/2}}{\alpha h^3} = Kh^{1/2}$$

which means

$$\frac{d}{ds}(P_1 - P_0) < kh^{1/2} - Kh^{1/2} < kh^{1/2}.$$

The second part of the lemma is proved perfectly analogously.

LEMMA 15. Let us assume that the lines $\Gamma_0 : y = y_0(x)$ and $\Gamma : y = y(x)$ have the following properties:

$$k_0\sigma < y(x) - y_0(x) < k\sigma,$$
$$|y_0'(x)| < \tau, \quad |y'(x)| < \tau, \tag{103}$$
$$|y_0''(x)| \leqslant \nu, \quad |y''(x)| \leqslant \nu,$$

where τ, ν; $\nu > \tau^2$ are arbitrarily small. Letting $\xi = f(z)$ ($f(\pm\infty) = \pm\infty$) denote a function which realizes the conformal mapping of the domain $D(\Gamma_0, D)$ onto the strip $0 < \eta < \sigma$ and assuming for points of $\Gamma: V(s) = |f'(z)|$, we obtain

$$|V(s_1 + \Delta s) - V(s_1)| < k\nu\Delta s \log\frac{1}{\Delta s} + k\frac{\tau}{s}\Delta s, \tag{104}$$

where ds is an element of arc of Γ and s_1 corresponds to the point z_1.

Proof. Performing a $1/\sigma$-fold similar expansion of the planes z and ξ, it can be seen that it is sufficient to examine the case when $\sigma = 1$. Noting this, let us draw an arc of the contour orthogonal to Γ and Γ_0 and located in the domain D, through the point s_1. Let us draw tangents L_0 and L, respectively, to Γ_0 and Γ. through the ends of this arc. In conformity with the condition of the lemma, the angle between the lines L_0 and L does not exceed 2τ. Let us map the dihedral L_0L conformally onto the unit strip $0 < \eta < 1$ such that the vertices of the angles would go over into the points $\pm\infty$: $\omega = \varphi(z)\,(e^{i\alpha})/\tau)\log(z-t_0)$, where t_0 is the vertex of the angle LL_0L; the distance between the point z_0 and s_1 will be $1/2\tau$. The segments of the curves Γ_0 and Γ which are in the strip $|x-x_1| \leq 1/4\tau$, $z_1 = x_1 + iy_1$ go over into the lines Γ_0' and Γ' under the mapping φ, where these lines will also satisfy the relationship (103) ($\sigma = 1$) for some other constants τ and ν by virtue of the condition $\nu = \tau^2$. Moreover, the lines Γ_0' and Γ' will go into the lines y = 0, y = 1, respectively, at the points $(1/\tau_0)\log|z_1 - z_0|$, $(1/\tau_0)\log|z_1 - z_0| + i$. Let $\overline{\Gamma}_0: y = \overline{y}_0(x)$ and $\overline{\Gamma}: y = \overline{y}(x)$ denote lines which coincide with Γ_0', Γ' at $|x-x_1| < 1/4\tau$ and satisfy condition (103) in the same sense as the lines Γ_0', Γ'. Let $\omega = \psi(z)$ map the domain $D(\Gamma_0, \Gamma)$ into $D(\overline{\Gamma}_0, \overline{\Gamma})$ under the condition of correspondence of the points z_1, $\omega_1 = (1/\tau_0)\log|z_1 - z_0| + i$ and the lines Γ_0, $\overline{\Gamma}_0$ and $\xi = f'(\omega)$ maps the domain $D(\overline{\Gamma}_0, \overline{\Gamma})$ onto the strip $0 < \eta < 1$, $f_1(\pm\infty) = \pm\infty$.

Hence, setting $V_1 = |f_1'|$, $v = |\varphi'|$, we obtain

$$f(z) = f_1[\psi(z)],$$
$$V(s) = |f_1'(\omega)|\,|\psi'(z)| = V_1 - v.$$

This means

$$|V(s+\Delta s) - V(s)| < k\,|V_1(s+\Delta s) - V_1(s)| + k\,|v(s+\Delta s) - v(s)|.$$

For the second member, for $|\Delta s| < 1/8\tau$

$$|v(s+\Delta s) - v(s)| < k\tau\Delta s.$$

By virtue of Theorem 10, for the first member we obtain

$$|V_1(s+\Delta s) - V_1(s)| < k\nu\Delta s \log\frac{1}{\Delta s},$$

for bounded values of Δs, $|\Delta s| < 1$, and after an elementary estimation of this same expression we obtain the estimate $k\tau\Delta s + k\nu$ for $\Delta s > 1/2$. Combining the inequalities obtained, we arrive at the desired relationship (104).

Let us return to the notation used at the beginning of this section and let us prove the lemma.

LEMMA 16. Let us assume that the functions $y_0(x)$ and $y_1(x)$ also satisfy the following conditions:

$$|y_0'(x)| < kh^{5/2}, \quad |y_0''(x)| < kh^2,$$
$$|y_1'(x)| < kh^{5/2}, \quad |y_1''(x)| < kh^2.$$

Moreover, let us have

$$|P_1 - P_0| < k, \tag{105}$$

along the line Γ_1, and let the lines Γ_1 contain the arc γ which has a differentiable curvature.

Under these conditions, if we have

$$P_1'(s_0) = kP_0'(s_0) + Q, \quad k > 0, \tag{106}$$

at a certain point of the arc, then $|P_0'(s_0)| < kh^{-1/2}$ at this point.

<u>Proof.</u> Let us assume the opposite, i.e., that $|P_0'(s_0)| > Kh^{-1/2}$ or for definiteness

$$P_0'(s_0) > Kh^{-1/2}, \tag{106'}$$

where K is arbitrarily large; in this case (106) will be equivalent to the condition

$$P_1'(s_0) = kP_0'(s_0). \tag{107}$$

By virtue of condition (106), there is a bounded number θ such that

$$|f_1'| = \theta\,|f_0'|.$$

at the point s_0. Hence, replacing the function f_0 by the function θf_0, at the point s_0 we have

$$P_1(s_0) = P_0(s_0). \tag{108}$$

Since P_0' hence does not vary, then without restricting the generality the condition (105) can be replaced by the condition (108) under the conditions of the lemma. Moreover, considering P_0 and P_1 as functions of ξ, η, by virtue of the boundedness of $|f_0'|$ and $|f_1'|$ the differentiation with respect to s in condition (107) and the desired inequality can be replaced by differentiation with respect to ξ. Noting this, let us henceforth consider that the point $i\alpha h^3$ corresponds to the point s_0 in the mappings f_0 and f_1.

Let us assume

$$s^+(\xi) = \int_0^{\xi} e^{-P_1(\xi, \alpha h^3)} d\xi, \quad s^-(\xi) = \int_0^{\xi} e^{-P_0(\xi, \alpha h^3)} d\xi.$$

By virtue of Lemma 15 we have

$$|s^+(\xi) - s - (\xi)| < \frac{k}{\alpha} h^{-\frac{1}{2}} (\xi^2 + |\xi|^{1+\delta}), \quad \delta > 0. \tag{109}$$

Now, let us represent the values of $dP_1/d\xi$ and $dP_0/d\xi$ in terms of the functions Q_0 and Q_1 at the point; we will have

$$\frac{\partial P_1}{\partial \xi} = \frac{\partial Q_1}{\partial \eta} = k \max \left| \frac{Q_1(t, 2\alpha h^3) - Q_1(\tau, \alpha h^3)}{\alpha h^3} \right| + \frac{\pi}{2} \frac{1}{\alpha^2 h^6} \int_{-\infty}^{\infty} \frac{Q_1(\xi, \alpha h^3) - Q_1(0, \alpha h^3)}{\operatorname{sh}^2 \frac{\pi}{2} \frac{\xi}{\alpha h^3}} d\xi,$$

$$\frac{\partial P_0}{\partial \xi} = \frac{\partial Q_0}{\partial \eta} = k \max \left| \frac{Q_0(t, 0) - Q_0(\tau, \alpha h^3)}{\alpha h^3} \right| + \frac{\pi}{2} \frac{1}{\alpha^2 h^6} \int_{-\infty}^{\infty} \frac{Q_0(\xi, \alpha h^3) - Q_0(0, \alpha h^3)}{\operatorname{sh}^2 \frac{\pi}{2} \frac{\xi}{\alpha h^3}} d\xi.$$

The first member in the expressions obtained for $\partial P_1/\partial\xi$ and $\partial P_0/\partial\xi$ is less than $kh^{-1/2}$ by virtue of the conditions of the lemma. In conformity with the assumption (106'), the sum of the remaining members should be greater than $Kh^{-1/2}$. Let us show that this sum is small together with h. Indeed, by introducing the variables s^+ and s^- the considered sum can be represented as

$$[Q_0(0, \alpha h^3)=Q_1(0, \alpha h^3),\ Q_0(s)=Q_1(s)];$$

$$\left|\frac{\pi}{2}\frac{1}{\alpha^2h^6}\int_{-\infty}^{\infty}\frac{Q[s^+(\xi)]-Q[s^-(\xi)]}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}}\right|<\frac{\pi}{2}\frac{1}{\alpha^2h^6}\int_{-\infty}^{\infty}\frac{|Q'|\,|s^+-s^-|\,d\xi}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}},$$

where Q (s) denotes the angle formed by the tangent to Γ_1 at the point s and the x axis. Remarking that $|Q'| < kh^2$ and substituting its majorant from (109) in place of $|s^+-s^-|$, we obtain that this last expression is small together with h. The lemma is thereby proved completely.

LEMMA 17. Under the conditions of the preceding lemma, let us assume that the curvature K(s) of the line Γ_1 does not exceed ν_1 and the line Γ_1 contains an arc γ of the contour of radius $1/\nu_1$.

Under these conditions there is an N = N (k) such that when

$$\nu_1=\nu_0+r\alpha h^3,\ r>N,$$

where ν_0 is the maximum of the curvature of the line Γ_0, we will have

$$P_1''(s_0)-P_0''(s_0)<-r+\left\{\left|\frac{\partial P_0}{\partial s}\right|+\left|\frac{\partial P_1}{\partial s}\right|\right\}\left|\frac{\partial P_1}{\partial s}-\frac{\partial P_0}{\partial s}\right|+\left(\frac{\partial P_1}{\partial s}\right)^2(e^{-P_1}-e^{-P_0}),$$

at any point s_0 of the arc γ if the arc γ is concave relative to D_0, $y_1'' < 0$, and

$$P_1''(s_0)-P_0''(s_0)>r-\left\{\left|\frac{\partial P_0}{\partial s}\right|+\left|\frac{\partial P_1}{\partial s}\right|\right\}\left|\frac{\partial P_1}{\partial s}-\frac{\partial P_0}{\partial s}\right|+\left(\frac{\partial P_1}{\partial s}\right)^2(e^{-P_1}-e^{-P_0}),$$

if the arc γ is convex relative to D_0, $y_1'' > 0$.

Proof. Let us consider the first part of the lemma. Let us assume that the point under consideration s_0 goes over into the point $i\alpha h^3$ in the mappings $\zeta = f_0(z)$, $\zeta = f_1(z)$. Let $Q_0'(\xi, \eta)$ and $Q'(\xi, \eta)$ denote the values of the derivatives of the functions Q_0 and Q_1 with respect to ξ. Let us find expressions for $\partial^2P_0/\partial\xi^2$ and $\partial^2P_1/\partial\xi^2$ at the point $i\alpha h^3$ in terms of the functions Q_0' and Q_1', respectively. Noting that for $\eta = 0$, αh^3

$$Q_0'=Ke^{-P_0}, \tag{110}$$

taking into account Lemma 15 and the inequality obtained,*

$$|P_0(\xi, \alpha h^3)-P_0(\xi, 0)|<\nu_1\alpha h^3<k\alpha h^5, \tag{111}$$
$$\frac{\partial P}{\partial\eta}=\frac{\partial Q}{\partial\xi}=Ke^{-P},$$

we have

$$\frac{\partial^2P_0}{\partial\xi^2}=\frac{\partial Q_0'}{\partial\eta}=k\frac{\nu_1-\nu_0}{\alpha h^3}+\frac{\pi}{2}\frac{1}{\alpha^2h^3}\int_{-\infty}^{\infty}\frac{Q_0'(\xi, \alpha h^3)-Q_0'(0, \alpha h^3)}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}}\,d\xi$$

or, considering the curvature K of the line Γ_1 as a function of ξ and using (110), we obtain

$$\frac{\partial^2P_0}{\partial\zeta^2}=r+\frac{\pi}{2}\frac{1}{\alpha^2h^6}\left(\int_{-\infty}^{\infty}\frac{[K(\xi)-K(0)]e^{-P_0}}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}}\,d\xi+\int_{-\infty}^{\infty}\frac{Q_0'(0, \alpha h^3)[e^{P_0(\xi)}-e^{-P_0(0)}]}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}}\right)d\xi. \tag{112}$$

Let us estimate the third member. To this end, let us consider that

$$Q_0'(0, \alpha h^3)=\frac{\partial P_0}{\partial\eta}=\frac{P_0(0, \alpha h^3)-P_0(0, 0)}{\alpha h^3}+\frac{\pi}{2}\frac{1}{\alpha^2h^6}\int_{-\infty}^{\infty}\frac{P_0(\xi)-P_0(0)}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}}\,d\xi+\frac{k}{\alpha^2h^6}\int_{-\infty}^{\infty}\frac{P_0(\xi, 0)-P_0(0, 0)}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}+k}\,d\xi.$$

*To obtain this inequality it is sufficient to use the relation $dP/d\eta = dQ/d\xi = Ke^{-P}$.

Hence, taking account of (112) and Lemma 14, we obtain

$$\left|\frac{1}{\alpha^2 h^6}\int_{-\infty}^{\infty}\frac{P_0(\xi)-P_0(0)}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}}\,d\xi\right|<\frac{k}{\alpha}\,h^{-1/2}.$$

Moreover, again by virtue of Lemma 14

$$\left|\int_{-\infty}^{\infty}\frac{[P_0(\xi)-P_0(0)]^n}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}}\,d\xi\right|<k^n h^{\frac{n}{2}}\int_{-\infty}^{\infty}\frac{|\xi|^n\,d(\xi)}{\operatorname{sh}^2\frac{\xi}{\alpha h^3}}<k^n\alpha^{n+1}h^{5/2n}.$$

Therefore,

$$\left|\frac{Q_0'(0,\alpha h^3)}{\alpha^2 h^6}\int_{-\infty}^{\infty}\frac{e^{-P_0(\xi)}-e^{P_0(0)}}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}}\,d\xi\right|<\frac{k}{\alpha}\,h^{3/2}.$$

Hence (112) yields

$$\frac{\partial^2 P_0}{\partial\xi^2}=r+\frac{\pi}{2}\frac{1}{\alpha^2 h^6}\int_{-\infty}^{\infty}\frac{[K_0(\xi)-K_0(0)]\,e^{-P_0(\xi)}}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{\alpha h^3}}\,d\xi. \tag{113}$$

Performing a completely analogous calculation for $\partial^2 P_1/\partial(\xi)^2$ at the very same point $0, \alpha h^3$, we obtain

$$\frac{\partial^2 P'}{\partial\xi^2}=k-\frac{\pi}{2}\frac{1}{h_1^2}\int_{-\infty}^{\infty}\frac{K_1(\xi)-K_1(0)}{\operatorname{sh}^2\frac{\pi}{2}\frac{\xi}{h_1}}\,d\xi. \tag{114}$$

Let us note that the integrand in (113) and (114) is not negative because the curvature of the line Γ_1 reaches its maximum on the arc γ by assumption; moreover, r is large compared to k

$$\left|\frac{\partial^2 P_1}{\partial\xi^2}\right|<k\left\{\frac{\partial^2 P_0}{\partial\xi^2}-\frac{\partial^2 P_1}{\partial\xi}\right\}. \tag{115}$$

Now, let us turn from the derivatives with respect to ξ to the derivatives with respect to s; we evidently have

$$\frac{\partial^2 P_0}{\partial s^2}=\frac{\partial^2 P_0}{\partial\xi^2}e^{2P_0}+\left(\frac{\partial P_0}{\partial\xi}\right)^2 e^{P_0}=\frac{\partial^2 P_0}{\partial\xi^2}e^{2P_0}+\left(\frac{\partial P_0}{\partial s}\right)^2 e^{-P_0},$$
$$\frac{\partial^2 P_1}{\partial s^2}=\frac{\partial^2 P_1}{\partial\xi^2}e^{2P_1}+\left(\frac{\partial P_1}{\partial\xi}\right)^2 e^{P_1}=\frac{\partial^2 P_1}{\partial\xi^2}e^{2P_1}+\left(\frac{\partial P_1}{\partial s}\right)^2 e^{-P_1}.$$

Hence, taking account of (113), (114), and (115), we obtain

$$\frac{\partial^2 P_1}{\partial s^2}-\frac{\partial^2 P_0}{\partial s^2}<\left(\frac{\partial^2 P_1}{\partial\xi^2}-\frac{\partial^2 P_0}{\partial\xi^2}\right)e^{2P_0}+\frac{\partial^2 P_1}{\partial\xi^2}\left(e^{2P_1}-e^{2P_0}\right)+$$
$$+\left\{\left(\frac{\partial P_1}{\partial s}\right)^2-\left(\frac{\partial P_0}{\partial s}\right)^2\right\}e^{-P_0}+\left(\frac{\partial P_1}{\partial s}\right)^2\left(e^{-P_1}-e^{P_0}\right)<$$
$$<-r+\left\{\left|\frac{\partial P_0}{\partial s}\right|+\left|\frac{\partial P_1}{\partial s}\right|\right\}\left|\frac{\partial P_1}{\partial s}-\frac{\partial P_0}{\partial s}\right|+\left(\frac{\partial P_1}{\partial s}\right)\left(e^{-P_1}-e^{P_0}\right). \tag{116}$$

The second part of the lemma is proved perfectly analogously.

LEMMA 18. In addition to the conditions of Lemma 17, let us assume that along Γ_1

$$|P_1(s)-P_0(s)|<\frac{kh}{\log\frac{1}{h}}. \tag{117}$$

Under these conditions, the function $q(s) = P_1(s) - P_0(s)$ cannot reach its maximum on the arc γ including its endpoints if γ is convex to D_0, and cannot reach its minimum if γ is concave to D_0.

Proof. Let us again examine the first case when γ is convex to D. Let us assume that we have

$$\frac{\partial P_1}{\partial s} = \frac{\partial P_0}{\partial s} + \frac{k\sqrt{h}}{\log\frac{1}{h}},$$

at an arbitrary point s_0 of the arc γ; however in this case in conformity with Lemma 15 we will have

$$\left|\frac{\partial P_1}{\partial s}\right| < kh^{-1/2}, \quad \left|\frac{\partial P_0}{\partial s}\right| < kh^{-1/2}.$$

In this case, the inequality (116) becomes

$$\frac{\partial^2 q}{\partial s^2} = \frac{\partial^2 P_1}{\partial s^2} - \frac{\partial^2 p_0}{\partial s^2} > r - \frac{k}{h}\left(e^{-P_1} - e^{P_0}\right) - \frac{k}{\log^2\frac{1}{h}}$$

or taking (117) into account, finally,

$$\frac{\partial^2 q}{\partial s^2} > \frac{1}{2} r.$$

Hence, these results directly show that the maximum cannot be achieved at any inner point of the arc γ. Moreover, it hence follows that when the maximum is achieved at the left endpoint of the arc γ, for instance, $s = s_1$, then either

$$\overline{\lim_{\Delta s \to +0}} \frac{q(s_1 + \Delta s) - q(s_1)}{\Delta s} < -\frac{k\sqrt{h}}{\log\frac{1}{h}},$$

or the right arbitrary function q will exist and will be negative at the point s_1, where in this case the function q will be convex to the right of s_1. According to Lemma 8 in [2], it can be shown in both cases that for $\Delta s < 0$

$$\lim \frac{q(s_1 + \Delta s) - q(s_1)}{\Delta s} < 0,$$

i.e., the maximum is not achieved at the point s_1. The case of the right endpoint of the arc γ is considered perfectly analogously.

12. EXISTENCE THEOREM

Using the auxiliary propositions proved above and applying a method analogous to the method developed in the problem of jet fluid flows, let us prove the existence of a solution of the problem formulated in Sec. 6. Let us prove the existence of waves with any sufficiently long period and let us obtain the solitary wave as a limit case.

THEOREM. For any sufficiently small h and sufficiently large ω there is always a value of Δ for which a solution $y = y(x, \omega)$ will exist for the equation

$$J(\gamma_0\Gamma) = |f'(z, \gamma_0, \Gamma)|^2 - C + \lambda y = 0,$$
$$C = 3 + 9h^2, \quad \lambda = \frac{2}{h} + 6\lambda,$$

where γ_0 is the line $y = \Delta(\omega)$, $|\Delta| < kh^{3}\log(1/h)$. The function $y = (x, \omega)$ will hence have the following properties:

1) $y(x, \omega)$ is a periodic function with period 2ω;

2) $y(x, \omega)$ is symmetric relative to the y axis;

3) $y(x, \omega)$ admits a single maximum at the point $x = 0$, $y(0, \infty) = h + \theta h^2$, $1 < \theta < 2$ in the segment $(-\omega, \omega)$.

For $\omega \to \infty$ the function $y(x, \omega)$ also tends to a solution of (1), where the value $y(x, \infty)$ which admits the single maximum at the point $x = 0$ and has the asymptote $y = \Delta$ is

$$y(0,\infty)=h+2h^2,$$
$$|\Delta(\infty)|<\frac{kh^3}{\log\frac{1}{h}}.$$

Proof. We perform the proof by induction by going from large values of v to smaller values. In conformity with this, let us first prove the existence of the solution to this problem:

a) Construct a solution of the equations

$$J_v(\Gamma_0,\Gamma)=0,\ v=Kh^2,$$

where $\Gamma_0 : y = y_0(x) = v + \eta(x)$ satisfies the conditions

$$y_0(-x)=y_0(x),\quad y_0(x+2\omega)=y_0(x),$$
$$|\eta(x)|<3h^3,\ |y_0'(x)|\leqslant h^{5/2},\ |y_0''(x)|\leqslant h^{3/2}. \tag{118}$$

Let us prove the existence of a solution of this problem for $K \geq 10$, where for this solution we will have

$$|h-y(x)|<2h. \tag{119}$$

Let us consider the family E of functions $\{y(x)\}$ which satisfy the following conditions:

$$|h-y(x)|<2h^2,\ y(-x)=y(x),\ y(x+2\omega)=y(x), \tag{120}$$
$$|y'(x)|\leqslant 1,\quad \lim\left|\frac{y'(x+\Delta x)-y'(x)}{\Delta(x)}\right|\leqslant\left[1+y'^2(x)\right]. \tag{121}$$

Let us define the functional $T(y)$ in this family:

$$T(y)=\max_{|x|\leqslant\omega}|J_v(\Gamma_0,\Gamma)|.$$

This functional is evidently continuous in E and the family E is compact. This means that a function $y_1(x)$ exists in E which yields the absolute minimum T in E. We must show that $T(y_1) = 0$. Let us assume the opposite, i.e., that $T(y_1) > 0$.

Hence, in order to arrive at a contradiction it is sufficient to show that $y_1(x)$ can be varied in E so that the appropriate variation of T would be negative. To this end, let us note some properties of the function

$$\varphi(x)=J_v(\Gamma_0',\Gamma),$$

where a line which maps the function $y = y(x)$ is taken as Γ_1.

1) At any point x where $y_1(x) = h + 2h^2$, $\varphi(x) < 0$ but where $y_1(x) = h-2h^2$, $\varphi(x) > 0$.

Indeed, by virtue of Theorem 1 ([2], p. 398), let us magnify $J_v(\Gamma_0, \Gamma_1)$ in the first case if we replace Γ_1 by the line $y = h + 2h^2$ and Γ_0 by the line $y = v + 8h^3$, i.e., we have at the considered point

$$\varphi=J_v(\Gamma_0,\Gamma_1)<\left(\frac{h-v}{h-v+2h^2-3h^3}\right)^2-C+\lambda(h+2h^2)<0.$$

The second case is considered completely analogously.

2) At points where $|\varphi(x)|$ achieves an absolute extremum, $|y_1'(x)| < 1$. This property results directly from the analysis performed in proving Lemma 3. Starting from calculations performed in proving Lemma 4 and using the results of Secs. 8 and 9 of this paper, it can be seen perfectly analogously that

3) If the curve Γ_1 contains an arc of the contour of unit radius, then the function φ cannot achieve its absolute maximum or minimum on this arc, including its endpoints, depending on whether $y_1'' < 0$ or $y_1'' > 0$ along this arc.

4) Let ε be an arbitrary small positive number and let the line $\Gamma_1 : y = \bar{y}_1(x)$ be such that

$$|\bar{y}_1(x) - y_1(x)| \leqslant \varepsilon$$

and at some point x_0

$$\bar{y}_1(x_0) = y_1(x_0) + \varepsilon, \quad [\bar{y}_1(x_0) = y_1(x_0) - \varepsilon].$$

It is asserted that under these conditions we have at the point x_0

$$J_v(\Gamma_0, \bar{\Gamma}_1) < J_v(\Gamma_0, \Gamma_1) \quad (J_v(\Gamma_0, \bar{\Gamma}_1) > J_v(\Gamma_0, \Gamma_1).$$

Indeed, by virtue of the Theorem 1 [2] mentioned, it is sufficient to prove our assertion for the case when $\bar{y}_1(x) = y_1(x) + \varepsilon \, [\bar{y}_1(x) = y_1(x) - \varepsilon]$ for any x; however, in this case

$$J_v(\Gamma_0, \Gamma_1) - J_v(\Gamma_0, \bar{\Gamma}_1) = (1 + \theta h)\frac{2\varepsilon}{h - v} - \frac{2\varepsilon}{h} - 6h\varepsilon,$$

where $|\theta| < 8$. On the other hand, substituting its least value in place of v, we finally obtain

$$J_v(\Gamma_0, \Gamma_1) - J_v(\Gamma_0, \bar{\Gamma}_1) > k\varepsilon, \quad k > 0. \tag{122}$$

Analogusly, if $\bar{y}_1(x) = y_1(x) - \varepsilon$, then

$$J_v(\Gamma_0, \Gamma_1) - J_v(\Gamma_0, \bar{\Gamma}_1) < -k\varepsilon. \tag{122'}$$

Taking account of the properties listed for the line Γ_1, as well as the evident continuity of the function φ, we can apply the construction given in Sec. 10 [2] of the theory of jets, and we can obtain a line $\bar{\Gamma}_1$ in the class E for which T takes on a smaller value than on Γ_1. The existence of the solution of the problem posed for $k \geq 10$ is thereby established. It follows directly from (122) and (122') that the solution of the problem posed is unique and depends continuously on $y_0(x)$.

b) Now, let us show that under the conditions a) there is a constant $\Delta = \Delta\{y_0\}$, $|\Delta| < h^2$ for sufficiently small h such that the solution of the equation

$$J_v(\Gamma_0', \Gamma) = 0, \tag{123}$$

where $\Gamma_0' : y = y_0(x) + \Delta$, will pass through the point $(0, h + \alpha)$, $0 \leq 2 \leq 2h^2$. Indeed, turning successively to (122) and (122') and assuming $\Delta = \pm (2/k)h^2$, we obtain the solutions of (123), one of which corresponds to the plus sign and will be greater than $h + 2h^2$ for $x = 0$; the other will correspond to the minus sign, and hence x will be less than h. We hence obtain the desired result from the continuity of the solution relative to $y_0(x)$.

c) Let us turn to realization of the induction. Let $Y = S\{y_0, v\}$ denote an integral of (11) (for $\eta = y_0(x) - v + C$ with period 2ω) which satisfies the initial conditions

$$Y(0) = h + \alpha, \quad Y'(0) = 0.$$

By virtue of Lemma 1, if $y_0(x)$ satisfies condition (118), a number k_0 exists such that for a sufficiently small variation of y_0 in comparison to h^3 we have

$$|S\{y_0+\delta y_0, v\} - S\{y_0, v\}| < \frac{k_0}{h} \max |\delta y_1(x)|. \quad (124)$$

By virtue of the preceding lemmas, under the very same assumptions there is a constant k such that for the function $y(x) = H\{y_0\}$ we will have

$$|y'(x)| \leqslant k_1 h^{5/2}, \ |y''(x)| \leqslant k_1 h. \quad (125)$$

If $y_0(x)$ satisfies the conditions of Lemma 13, there exist a θ_0 and k_2 such that

$$|s\{y_0, v\} - H\{y_0\}| < \frac{k_2 h^2}{\log \frac{1}{h}}. \quad (126)$$

Now, let us note one special construction. Starting from $y_0(x)$ which satisfies (118), let us construct a function $y_1(x)$ such that $S\{y_1, v + kh^3\} = S\{y_0, v\}$, $k \leq 1$.

Let us assume that a function $H\{y_1\} = y(x)$ and an appropriate functional $\Delta\{y_1\}$ can be constructed for the function $y_1(x)$. Let $\Gamma_0 : y = y_0(x) + \Delta\{y_1\}$ and $\Gamma_1 : y_1(x) + \Delta\{y_1\}$. For the domains $D_0(\Gamma_0, \Gamma_1)$ and $D_1(\Gamma_1 \Gamma)$, $\Gamma : y = y(x)$ let us construct the mappings f_0 and f_1 introduced in Sec. 11. Starting from (126), it can be shown that we will have

$$|P_1 - P_0| < \frac{2k_2 h}{\log \frac{1}{h}} \quad (127)$$

along the line Γ_1.

Now let $y_0(x)$ satisfy conditions (118) and (89) for $\nu = \nu_0$, where ν_0 is so small that the result of Lemma 11 holds for $y_0(x)$:

$$|\delta H\{y_0\}| < \frac{k_2}{h^2} \max |\delta y_0|. \quad (128)$$

Let us assume

$$\bar{r} = h^3/4k_3 = \beta h^3,$$

and let n be the least integer greater than $10h^2/\bar{r} = 10/\beta h$. Let us separate the interval $(0, 10h^2)$ into n equal parts, where $v_1, v_2, \ldots, v_{n-1}$ are the points dividing $v_m = 10h^2$, $m/n = m\tau_1$. Let $Y_0^{(m)}(x)$ denote a function such that $S\{Y_0^{(m)}, v_m\} = Y(x)$, where $y = Y(x)$ is the Rayleigh wave equation. Let us construct some neighborhood for each line $\Gamma_1^m : y = Y_0^m(x)$.

Setting $\delta_0 = 0$, we denote the numbers $\delta_1, \delta_2, \ldots, \delta_k$ from the auxiliary relation

$$\delta_m = \delta_{m-1}(1 + 2k_0 \beta h) + \frac{2k_2 h^4}{\log \frac{1}{h}}. \quad (129)$$

It hence follows that

$$\delta_n \leqslant \frac{k_4 h^3}{\log \frac{1}{h}}, \quad (130)$$

where k_4 is a number defined by the numbers β, k_0 and k_2. Let us henceforth consider k so small that δ_n would satisfy the inequality $\delta_m < \theta_0$.

In addition to the numbers δ_m, let us still introduce the numbers δ'_m and δ''_m. Let us set $\delta'_m = k_5 h^{7/2} m$, where k_5 is a number defined in terms of k_1 by using the function $\bar{K}_0(k)$ introduced in Lemma 14:

$$k_5 = \bar{K}_0(k_1).$$

Similarly

$$\delta''_m = k_6 h_m^3,$$

where the number k_6 is also expressed in terms of k_1 by using the function $K_1(k)$ from Lemma 17.

Let E_m denote the set of lines $\{\gamma_0^{(m)}\}$: $\{y = y_0(x, m\}$ which have the properties

$$\begin{gathered}\operatorname{var}|y_0(x, m) - Y_0^{(m)}(x)| \leqslant \delta_m < \delta_n,\\ y_0(-x, m) = y_0(x, m), \quad y_0(x+2\omega) = y_0(x, m);\end{gathered} \tag{131}$$

$$\begin{gathered}|y_0'(x, m)| \leqslant \delta'_m,\\ |y_0''(x, m)| \leqslant \delta''_m;\end{gathered} \tag{132}$$

moreover, especially for the line $y_0(x, n)$,

$$|y_0(x, n) - Y_0^{(n)}(x)| \leqslant vh^3. \tag{133}$$

Let us establish the existence of the solution $H\{y_0(x, m)\}$, by induction from m to $m-1$, such that the line $\eta = v_n$ will go over into a line of the family E_n for the mapping $f_1^{(m)}$ of the strip $v_m < \eta < h$ into the domain $D(\gamma_0^{(m)})$, $H\{y_0(x, m)\} > y > y_0(x, m)$. By virtue of a), this assertion holds for $m = n$. Let us assume it to be valid for m and let us prove it for $m-1$.

Let $y_0(x, m-1)$ be an arbitrary line of the family E_{m-1}. By virtue of (131), (124), and Lemma 1, we will have

$$|s\{y_0, v_{m-1}\} - Y(x, \alpha)| < \frac{k_0\delta_{m-1}}{h}. \tag{134}$$

Let us determine $y_1(x)$ from the condition

$$s\{y_1, v_m\} = s\{y_0, v_{m-1}\}.$$

It can be seen that

$$\operatorname{var}|y_1 - Y_0^{(m)}(x)| < (1 + 2k_0\beta h)\operatorname{var}|y_0(x, m-1) - Y_0^{m-1}(x)|. \tag{135}$$

This means that the function $y_1(x)$ belongs to the family E_m, where by virtue of (126) we have

$$|s\{y_1, v_m\} - H\{y_1\}| < \frac{k_2h^2}{\log\frac{1}{h}}. \tag{136}$$

It is hence assumed that $H\{y_1\}$ exists and the line $f_1^{(m)}$ corresponds to some line of the family E_n for the mapping $f_1^{(m)}$.

Now, let $y = \bar{y}_0(x) = y_0(x, m-1) + C$, $C = \text{const}$ be a function such that for $\eta = \bar{y}_0(x) - v_{m-1}$ and for $v = v_{m-1}$ the integral of (11) agrees with $s\{y_0(x, m-1)\}$ and $y = \bar{y}_1(x)$ is such that for $\eta = \bar{y}_1(x) - \delta_m$ and for $v - v_m$ the integral of (11) agrees with $s\{y_0(x, m-1)\}$ also.

Let us perform the conformal mapping $\zeta = \bar{f}_0(z)$, $\zeta = \bar{f}_1(z)$ of the strips $\bar{y}_0(x) < y < \bar{y}_1(x)$ and $\bar{y}_1(x) < y < H\{y_1\}$ respectively, onto the strips $v_{m-1} < \eta < v_m$ and $vm < \eta < h$. By virtue of (136), (127) we will have

$$|P_1 - P_0| < \frac{2k_2h}{\log\frac{1}{h}} \tag{137}$$

along the line $y = \bar{y}_1(x)$.

Noting this, let us construct a family of lines $F = \{\Gamma_1\} : \{y = y(x)\}$ in the neighborhood of the line $y = \bar{y}_1(x)$, with the following properties:

$$|y(x) - \bar{y}_1(x)| \leqslant \frac{k_2 h^4}{\log \frac{1}{h}},$$
$$y(-x) = y(x), \quad y(x+2\omega) = y(x),$$
$$|y'(x)| \leqslant \delta'_m, \quad |y''(x)| \leqslant \delta''_m.$$

All the lines of the family F belong to the family E_m, which means that for each line from F there exists an $H\{y\}$ and $\Delta\{y\}$. Let us extract a part F' out of F which is defined by the inequality

$$|\Delta\{y\} - \Delta\{\bar{y}\}| < \frac{kh^3}{\log \frac{1}{h}}. \tag{138}$$

Let $\zeta = f_0(z)$, $\zeta = f_1(z)$ realize the conformal mapping of the domains $y_0(x) + \Delta\{y\} < y < y(x) + \Delta\{y\}$ and $y\{x\} + \Delta\{y\} < y < H\{y\}$, respectively, onto the strips $v_{m-1} < \eta < v_m$ and $v_m < \eta < h$ under the condition of correspondence of the infinitely remote points. Let us introduce the function

$$\psi(s, \Gamma_1) = P_1 - P_0 = \log|f_1'(z)| - \log|f_0'(z)|$$

on each line $\Gamma_1 : y = y(x)$ F' by using these mappings and let us set

$$J(\Gamma) = \max_{|s| < \infty} \psi(s, \Gamma).$$

There remains to show that there exists a line Γ in the class of lines F' for which $J(\Gamma) = 0$. Let us assume the opposite, i.e., that the minimum of J in F' is positive,

$$\inf J(\Gamma) = a > 0.$$

By virtue of the compactness of F' this minimum is reached on some line $\Gamma^{(0)}$ of this same family,

$$J(\Gamma^{(0)}) = a.$$

Moreover, the line $y = \bar{y}_1(x)$ belongs to F' by virtue of (137),

$$a \leqslant \frac{2k_2 h}{\log \frac{1}{h}}.$$

It must be shown that $\Gamma^{(0)}$ in the class F' can be varied so that $\delta J(\Gamma^{(0)}) < 0$. To do this, let us note the following properties of the line $\Gamma^{(0)}$ and the function $\psi(x) = \psi(x, \Gamma^{(0)})$: 1) by virtue of (128) and the choice of τ at points where δy will reach the absolute maximum ($\delta y > 0$) and minimum ($\delta y < 0$), we will have $\delta\psi > 0$ ($\delta\psi < 0$); 2) if

$$\operatorname{var} |\bar{y}(x) - \bar{y}_1(x)| < \frac{2k_2 h^4}{\log \frac{1}{h}},$$

then at points where $y(x) - \bar{y}_1(x)$ will reach the maximum (minimum), we will have $\psi < a$ ($\psi > a$); 3) analogous inequalities hold if the inequality (138) will become an equality at appropriate points; 4) by virtue of Lemmas 16 and 18 the function ψ cannot reach the absolute minimum (maximum) at points where $y' = \delta'_m$ ($y' = -\delta'_m$) and on arcs of the greatest convexity

$$y'' = \delta''_m (1 + y'^2)^{3/2}, \quad y'' = -\delta''_m (1 + y'^2)^{3/2}.$$

However, the four properties listed for the line $\Gamma^{(0)}$ are sufficient for application of the variation $\delta\Gamma^{(0)}$ given in the paper about jets ([2], pp. 431-436) to the construction.

The formulated theorem is thereby proved completely.

13. SUMMARY

1. As is known, the problem of plane steady motion of an ideal gravity fluid in a channel of finite depth is equivalent to the following boundary-value problem from conformal mapping theory:

Let $D(\Gamma)$ be a domain in the plane of the complex variable $z = x + iy$ bounded by the real axis x and the curve $\Gamma: y = y(x)$, $y(x) > 0$. Furthermore, let $w = f(z, \Gamma)$, $f(\pm\infty, \Gamma) = \pm\infty$ be a function performing the conformal mapping of the domain $D(\Gamma)$ onto the strip $0 < v < h$ of the $w = u + iv$ plane. For given constants C, λ and h, find Γ such that the relationship

$$I(\Gamma)=|f'(z, \Gamma)|^2 - C+\lambda y=0, \ C>0, \ \lambda>0 \tag{1}$$

would hold at each point of Γ.

If it is assumed in addition that the desired function $y(x)$ differs slightly from a constant, and its derivatives are small and the boundary condition (1) is linearized in conformity with this, then the problem posed will admit of elementary solution, and the desired function will be a sinusoid with arbitrary amplitudes and phase and with period governed by the given constants.

A number of investigations have appeared in the past 20 years, in which a rigorous solution of the problem has been given by using integral equations, for cases slightly different from the linear case.

In addition, Rayleigh gave an approximate method for the case of waves in channels of low depth by taking account of the quadratic term. The Rayleigh theory afforded the possibility of examining waves, radically different from sinusoids; in particular, the Rayleigh theory gave the solution of the problem in the form of a line with a single maximum point (a solitary wave).

A number of propositions referring to the rigorous theory of almost Rayleigh waves is established in this paper. Underlying the method are general boundary properties of univalent functions which the author had used earlier to construct a qualitative theory of jet fluid motions.

2. In conformity with the conditions for which the Rayleigh solution can be considered as an approximate solution, let us assume that the number h is sufficiently small and the numbers C and λ have the structure

$$\lambda = \frac{2}{h} + (6+\alpha)h, \quad C = 3 + (9+\beta)h^2,$$

where α and β are sufficiently small quantities.

Henceforth, $k_1, k_2, \ldots$, will denote constants independent of h.

Under these conditions a general existence theorem holds.

THEOREM 1. For all values of $\omega > k_1\sqrt{h}$, where k_1 is sufficiently large, there exists a curve $\Gamma\omega: y = y(x, \omega)$ with period 2ω

$$y(x+2\omega, \omega)=y(x, \omega)$$

and with vertex at $x = 0$ which satisfies the functional equation (1).

The limit line $\Gamma_\omega: y(x, \omega)$ exists as $\omega \to \infty$ and yields an aperiodic solution of (1) with a single vertex at the point $x = 0$. This limit solution is an identity wave.

3. The connection between the solution $y(x, \omega)$ of Theorem 1 and the approximate solution $Y(x, \omega)$, $Y(x + 2\omega, \omega) = Y(x, \omega)$ given by Rayleigh is established by the following propositions.

THEOREM 2. Under the conditions taken in Sec. 2, we have for the solution $y(x, \omega)$

$$|y'(x, \omega)| < k_2 h^{3/2},$$
$$|y''(x, \omega)| < k_3 h,$$
$$|y'''(x, \omega)| < \frac{k_4}{\log \frac{1}{h}}.$$

THEOREM 3. Under the previous conditions, the estimate

$$|y(x, \omega) - Y(x, \omega)| < \frac{k_5 h^2}{\log \frac{1}{h}}$$

holds.

4. The following stability for the solution is essential to an algorithmic construction of the solution $y(x, \omega)$.

THEOREM 4. If the line $\gamma : y = \varphi(x)$

$$\varphi(x+2\omega) = \varphi(x), \quad \varphi(0) = y(0, \omega)$$

with vertex at the point $x = 0$ deflects from the solution $\Gamma : y = y(x, \omega)$ by more than ε, $\varepsilon > 0$

$$\sup |\varphi(x) - y(x, \omega)| \geqslant \varepsilon,$$

then $|I(\gamma)| > k_6 \varepsilon$.

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